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CALCULATION OF SUPERSONIC FLOWS AT LARGE DISTANCES FROM SLENDER LIFTING BODIES

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16. Abstract

The exact gasdynamical equations are solved far from the axis of a slender body flying at an angle of incidence. The boundary conditions are obtained from slender-body theory. The solution allows one to make a prediction of the strength of the sonic boom as well as the position of the shock waves.

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CALCULATION OF SUPERSONIC FLOWS AT LARGE DISTANCES FROM SLENDER LIFTING BODIES*

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SUMMARY

The exact gasdynamical equations are solved far from the axis of a slender body flying at an angle of incidence. The boundary conditions are obtained from slender-body theory. The solution allows one to make a prediction of the strength of the sonic boom as well as the position of the shock waves.

INTRODUCTION

Several theories have been developed for calculating sonic-boom ground pressure patterns and signatures. Most of these theories, such as those of Lansing (ref. 1), Hayes, Haefeli, and Kulsrud (ref. 2), and Friedman, Kane, and Sigalla (ref. 3), are based on geometric acoustics. These theories have great advantages in calculating sonic booms for unsteady flight conditions in an inhomogeneous atmosphere. Although corrections have been made by introducing nonlinear distortions, these theories are essentially linear in nature and are not adequate for handling such nonlinear effects as shock focusing or cut-off Mach number. On the other hand, Whitham's theory (ref. 4) for the supersonic flow about noninclined axisymmetric bodies is based on a modified theory of characteristics. The results are obtained as solutions of the nonlinear differential equations of supersonic flow.

The purpose of the present paper is to develop a theory to determine the supersonic flow in the far field of a lifting body with a nearly circular cross section. The nonlinear differential equations for supersonic flow are systematically expanded in a perturbation series. The resulting system of differential equations can be easily integrated for each order of magnitude.

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The flow of a perfect, nonviscous gas far away from a body flying at an angle of attack with a Mach number greater than 1 is considered. The body must be tapered, slender, and have a nearly circular cross section. Discontinuities in slope are allowed: however, abrupt changes in cross-sectional area (e.g., by a stabilizer) are not permitted. Small amplitude oscillations around an axis normal to the direction of flight and through the center of gravity, as well as effects of an inhomogeneous atmosphere, can be included but are not considered in this paper. The extension to a stratified atmosphere is straightforward. The extension would allow an investigation of the nonlinear effects at the caustic. The flow is assumed to be steady, homentropic, and homoenergetic. There are no heat or mass sources in the flow. These conditions imply that the flow is irrotational.

SYMBOLS

A cross-sectional area of body, πK^2

a speed of sound

C_p pressure coefficient

$c_1 = 1 - \frac{n+1}{n} \frac{1}{\cos^2 \mu^{(0)}}$

$c_2 = -2 \frac{n+1}{n} \frac{\sin \mu^{(0)}}{\cos^3 \mu^{(0)}}$

$D_\xi = \frac{\partial}{\partial \xi} = \partial_\xi$

$D_r = \frac{\partial x}{\partial \xi} \frac{\partial}{\partial r} - \frac{\partial x}{\partial r} \frac{\partial}{\partial \xi} = x_\xi \, \partial_r - x_r \, \partial_\xi$

$D_\psi = \frac{\partial x}{\partial \xi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \xi} = x_\xi \, \partial_\psi - x_\psi \, \partial_\xi$

$\left.\begin{array}{l} \vec{e}^1 = \text{grad } x^1 \\ \vec{e}^2 = \text{grad } x^2 \\ \vec{e}^3 = \text{grad } x^3 \end{array}\right\}$ Cartesian base vectors

f_m function of source distribution

$G_m^{(j)}(\xi)$ function of integration

H	total enthalpy
h	specific enthalpy
$K(x,\psi)$	radius of body
M	Mach number
m	Fourier number
$\vec{m}$	generatrix of Monge (or Mach) cone
n	number of degrees of freedom of the gas
$\vec{n}$	generatrix of wave normal cone
$\vec{n}_\infty$	generatrix of wave normal cone in undisturbed flow
$\vec{n}_\xi$	generatrix of wave normal cone in disturbed flow
$\vec{n}_S$	normal to shock surface
p	pressure
R	distance away from the body at which the boundary condition is satisfied
r	cylindrical variable (radial distance)
s	entropy
$\vec{s}$	vector of the space $\tilde{E}^3$
T	temperature
t	time
$\vec{t}$	vector of the space $\tilde{E}^3$
u_∞	velocity of undisturbed flow field

w	magnitude of the speed, $\lvert\vec{w}\rvert$
$\vec{w}$	vector of the speed
$\vec{w}_i$	components of the vector of the speed in a locally dependent base $(i = 1,2,3)$
x	abscissa of characteristic surface
x_s	abscissa of shock front

$$\left.\begin{aligned} x^1 &= x \\ x^2 &= r \cos \psi \\ x^3 &= r \sin \psi \end{aligned}\right\} \quad \text{coordinates in the space } E^3$$

$$\left.\begin{aligned} \vec{y}^1 &= \operatorname{grad} \xi \\ \vec{y}^2 &= \operatorname{grad} r = \vec{s} \\ \vec{y}^3 &= \operatorname{grad} \psi = \frac{1}{r} \vec{t} \end{aligned}\right\} \quad \text{vectors in the space } Y^3$$

z	integration variable
α	angle of attack
β	Prandtl factor, $\sqrt{M^2 - 1}$
γ	ratio of specific heats
η	Prandtl transformation, βr
ϑ	angle of inclination of the streamlines
$\lambda, \kappa_1, \kappa_2, \kappa_3$	factors of proportionality
μ	Mach angle
ξ	characteristic variable
ρ	density

σ angle between shock front and undisturbed flow

$\upsilon^i = (w, \vartheta, \varphi + \psi)$ $(i = 1,2,3)$

φ azimuth angle

ϕ potential function

ψ cylindrical variable (azimuth angle)

ω Prandtl-Meyer angle

Subscripts:

∞ state of undisturbed flow

$*$ critical values

Harmonic (or Fourier) numbers are denoted by numbers in the subscript position or by the subscript m. The order of magnitude is indicated by positive numbers in parentheses in the superscript position or by the superscript (j). The Greek letters μ, ν, σ, and τ used as subscripts and superscripts denote the Einstein summation convention. Single, double, and triple primes denote first, second, and third derivatives, respectively. A comma preceding a subscript denotes differentiation with respect to that subscript.

ANALYSIS

The exact nonlinear system of partial differential equations for supersonic flow is solved for large distances by using a perturbation method developed by Poincaré, Lighthill, and Kuo (see ref. 5).

The basic differential equations of the problem – the equations of continuity, irrotationality, and the condition of the characteristic surface – are expressed in invariant form. At first, this set of equations is expressed in cylindrical coordinates (x, r, ψ). It is advantageous to express the velocity vector $\vec{w}$ in a particular locally dependent basis and to use the characteristic variable ξ in place of the independent variable x. Consequently, the differential equations are transformed to ξ, r, and ψ as independent variables and to the magnitude w, the angle of inclination ϑ, and the azimuth angle φ of the velocity vector as dependent variables. The nonlinear equations obtained by this procedure are split into systems of quasi-linear differential equations of different orders of magnitude by expanding each unknown function in a perturbation series. These systems of equations can be integrated easily in each order of magnitude. The integrations lead to arbitrary functions of integration which are determined by the boundary condition calculated from slender-body theory. All physical properties of the far flow field can be calculated by this method. By using the three-dimensional analogy of Pfriem's formula (see ref. 6), the front shock can be calculated as well. Pfriem's formula for two dimensions states that for weak shocks the shock wave bisects the angle between the characteristics before and behind the shock.

Formulation of Basic Equations

The following equations, written in their invariant forms, describe the problem. Equation (1) gives the condition of irrotationality; equation (2), the condition of continuity; and equation (6), the condition of the characteristic surface.

Condition of irrotationality.- The condition of irrotationality is

$$\operatorname{curl} \vec{w} = 0 \tag{1}$$

Condition of continuity.- The condition of continuity is given as

$$\frac{d\rho}{dt} + \rho \operatorname{div} \vec{w} = 0 \tag{2}$$

By the second law of thermodynamics $\left(T\, ds = dh - \frac{1}{\rho}\, dp\right)$, the conditions of homentropic and homoenergetic flow are

$$\frac{ds}{dt} = 0$$

and

$$H = h + \frac{1}{2} w^2$$

The condition of continuity can be rewritten as

$$a^2 \operatorname{div} \vec{w} - w\vec{w} \cdot \operatorname{grad} w = 0$$

in which $a^2 = \left.\frac{dp}{d\rho}\right|_s$ is the definition of the speed of sound.

Condition of characteristic surface. - According to the theory of characteristics the generatrix of the wave normal cone $\vec{n}$ is normal to the generatrix of the Monge cone $\vec{m}$. The quantity ξ denotes the variable along the generatrix $\vec{n}$. Thus, the generatrix $\vec{n}$ is proportional to grad ξ; that is, $\vec{m} \propto \text{grad}\ \xi$ or, with λ used as the factor of proportionality,

$$\lambda\vec{n} = \text{grad}\ \xi \tag{3}$$

The speed of sound a is related to the velocity $\vec{w}$ by the relation

$$\vec{w} \cdot (-\vec{n}) = a \tag{4}$$

As $\vec{n}$ is a unit vector ($\vec{n} \cdot \vec{n} = 1$), equation (3) yields

$$\lambda^2 = \text{grad}\ \xi \cdot \text{grad}\ \xi \tag{5}$$

Substituting from equations (4) and (5) into equation (3) gives the equation of the characteristic surface in invariant form as

$$a^2 \text{grad}\ \xi \cdot \text{grad}\ \xi = (\vec{w} \cdot \text{grad}\ \xi)^2 \tag{6}$$

The interrelation between the Monge cone and the wave normal cone is shown in figure 1. The Mach angle is denoted by μ.

System of equations. - Within the system of basic equations composed of equations (1), (2), and (6), there are five scalar differential equations for determining the unknowns ξ and $\vec{w}$. Since the flow is homentropic, there exists the relation $p \propto \rho^\gamma$ between pressure and density where γ is the ratio of specific heats. Hence,

$$h = \frac{1}{2} n a^2 + \text{Constant}$$

where $n = \frac{2}{\gamma - 1}$ denotes the number of degrees of freedom of the gas. Combining this equation with the equation for homoenergetic flow gives

$$n a^2 + w^2 = \text{Constant}$$

Thus, the speed of sound a is a function of the velocity $\vec{w}$.

If one thinks of equation (6) as the defining equation for ξ, then there are four other equations (eqs. (1) and (2)) for determining the vector of speed $\vec{w}$. Though the three equations $\operatorname{curl} \vec{w} = 0$ are dependent on one another, they must be used together with the condition of continuity (eq. (2)). Alone, $\operatorname{curl} \vec{w} = 0$ gives the solution $\vec{w} = \operatorname{grad} \phi$, as $\operatorname{curl} \operatorname{grad} \phi \equiv 0$.

Change to a Characteristic Coordinate System

After having formulated the system of equations in an invariant form, one must select a coordinate system in which to carry out the solution. A plotted point P has the coordinates

$$x^1 = x$$

$$x^2 = r \cos \psi$$

and

$$x^3 = r \sin \psi$$

as indicated in figure 2.

The unit vectors in the space E^3 are

$$\left.\begin{aligned} \vec{e}^1 &= \operatorname{grad} x^1 \\ \vec{e}^2 &= \operatorname{grad} x^2 \\ \vec{e}^3 &= \operatorname{grad} x^3 \end{aligned}\right\} \tag{7}$$

as shown in figure 2. From equation (7) one calculates

$$\left.\begin{aligned} \operatorname{grad} r &= \vec{s} = \cos \psi \vec{e}^2 + \sin \psi \vec{e}^3 \\ \operatorname{grad} \psi &= \frac{1}{r} \vec{t} = \frac{1}{r}\left(-\sin \psi \vec{e}^2 + \cos \psi \vec{e}^3\right) \end{aligned}\right\} \tag{8}$$

As shown in figure 3, $\vec{e}^1, \vec{s}, \vec{t}$ is also a system of orthogonal vectors. It is useful to express the velocity vector $\vec{w}$ in a locally dependent basis and to define a new set of velocity components in this basis. The space $\tilde{E}^3\left(\vec{e}^1, \vec{s}, \vec{t}\right)$ just introduced will be used. According to figure 3 the velocity vector can be related to its magnitude w, its angle of inclination ϑ, and its azimuth angle $\varphi + \psi$. So the velocity vector is written as

$$\vec{w} = w\left[\cos\vartheta\vec{e}^1 + \sin\vartheta(\cos\varphi\vec{s} + \sin\varphi\vec{t})\right] \tag{9}$$

In the space $\tilde{E}^3(\vec{e}^1,\vec{s},\vec{t})$ the new components of velocity $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ are defined as

$$\vec{w}_i = \frac{\partial\vec{w}}{\partial\upsilon^i} \qquad (i = 1,2,3) \tag{10}$$

where

$$\upsilon^i = (w,\vartheta,\varphi + \psi) \qquad (i = 1,2,3)$$

So far, the unknown functions have been expressed in terms of cylindrical coordinates; that is,

$$\vec{w} = \vec{w}(x,r,\psi)$$

and

$$\xi = \xi(x,r,\psi)$$

Since disturbances sent out by a body travel along characteristic surfaces, it is preferable to introduce a "characteristic variable" ξ along the generatrix $\vec{n}$. Hence, the independent variable x is replaced by the formerly dependent variable ξ. This step follows the Poincaré-Lighthill-Kuo (P.L.K.) method (see ref. 5).

The new independent variable defines those characteristic surfaces propagated from the body. Disturbances traveling along the other family of characteristic surfaces are of lower intensity. For this reason, and also because the coordinate r seems to be physically much more reasonable, it is convenient to retain the variable r and avoid introducing a second characteristic variable. Now

$$\vec{w} = \vec{w}(\xi,r,\psi)$$

and

$$x = x(\xi,r,\psi)$$

Final Form of Equations

The base vectors $\vec{y}^1$, $\vec{y}^2$, and $\vec{y}^3$ for the coordinate system ξ, r, ψ are defined by the equations

$$\left.\begin{aligned} \vec{y}^1 &= \text{grad } \xi \\ \vec{y}^2 &= \text{grad } r = \vec{s} \\ \vec{y}^3 &= \text{grad } \psi = \frac{1}{r}\vec{t} \end{aligned}\right\} \tag{11}$$

Although $\vec{y}^2$ is a unit vector, $\vec{y}^1$ and $\vec{y}^3$ are not.

In the space $Y^3(\vec{y}^1, \vec{y}^2, \vec{y}^3)$,

$$\vec{e}^1 = \text{grad } x^1 = \vec{e}^\mu \frac{\partial x^1}{\partial \xi^\sigma} \frac{\partial \xi^\sigma}{\partial x^\mu} = \text{grad } \xi^\sigma \frac{\partial x^1}{\partial \xi^\sigma} = x_\xi \vec{y}^1 + x_r \vec{y}^2 + x_\psi \vec{y}^3 \tag{12}$$

Here and in the following equations, the Einstein summation convention is used. The Greek subscripts and superscripts μ, σ, and τ indicate this summation convention.

From equation (12) one determines that

$$\vec{y}^1 = \frac{1}{x_\xi}\left(\vec{e}^1 - x_r \vec{s} - \frac{1}{r} x_\psi \vec{t}\right) \tag{13}$$

Analogously, the operator grad is determined in the space Y^3 as

$$x_\xi \text{ grad} = \vec{e}^1 D_\xi + \vec{s} D_r + \frac{1}{r}\vec{t} D_\psi \tag{14}$$

where D_ξ, D_r, and D_ψ are abbreviations for the operators defined by the equations

$$D_\xi = \partial_\xi \qquad D_r = x_\xi \partial_r - x_r \partial_\xi \qquad D_\psi = x_\xi \partial_\psi - x_\psi \partial_\xi$$

Furthermore,

$$\left.\begin{aligned} \text{curl } \vec{w} &= \vec{e}^\tau \times \frac{\partial \vec{w}}{\partial \upsilon^\sigma} \frac{\partial \upsilon^\sigma}{\partial x^\tau} = \text{grad } \upsilon^\sigma \times \vec{w}_\sigma \\ \text{div } \vec{w} &= \vec{e}^\tau \cdot \frac{\partial \vec{w}}{\partial \upsilon^\sigma} \frac{\partial \upsilon^\sigma}{\partial x^\tau} = \text{grad } \upsilon^\sigma \cdot \vec{w}_\sigma \end{aligned}\right\} \tag{15}$$

By using equations (15), equations (1), (2), and (6) can be written in terms of the velocity components, $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ and the independent variables ξ, r, and ψ. The results are

$$\left.\begin{aligned} &\vec{w}_1 \times \text{grad } w + \vec{w}_2 \times \text{grad } \vartheta + \vec{w}_3 \times \text{grad}(\varphi + \psi) = 0 \\ &-\cot^2\mu\, \vec{w}_1 \cdot \text{grad } w + \vec{w}_2 \cdot \text{grad } \vartheta + \vec{w}_3 \cdot \text{grad}(\varphi + \psi) = 0 \\ &(\vec{w} \cdot \text{grad } \xi)^2 = a^2 \text{grad } \xi \cdot \text{grad } \xi \end{aligned}\right\} \qquad (16)$$

To simplify equations (16) the Prandtl-Meyer function $d\omega = \cot \mu \frac{dw}{w}$ is introduced. Here, ω denotes the Prandtl-Meyer angle and μ denotes the Mach angle. By using equation (10) the vectors $\vec{w}_i$ can be expressed in terms of the orthogonal vectors $\vec{e}^1$, $\vec{s}$, and $\vec{t}$. The five scalar differential equations obtained by this procedure are

Condition of irrotationality:

In direction of $\vec{e}^1$,

$$\tan \mu \sin \vartheta \sin \varphi D_r\omega - \frac{1}{r} \tan \mu \sin \vartheta \cos \varphi D_\psi\omega + \cos \vartheta \sin \varphi D_r\vartheta$$

$$- \frac{1}{r} \cos \vartheta \cos \varphi D_\psi\vartheta + \sin \vartheta \cos \varphi D_r\varphi + \frac{1}{r} \sin \vartheta \sin \varphi D_\psi\varphi + \frac{1}{r} \sin \vartheta \sin \varphi x_\xi = 0 \qquad (17a)$$

In direction of $\vec{s}$,

$$\tan \mu \sin \vartheta \sin \varphi D_\xi\omega - \frac{1}{r} \tan \mu \cos \vartheta D_\psi\omega + \cos \vartheta \sin \varphi D_\xi\vartheta$$

$$+ \frac{1}{r} \sin \vartheta D_\psi\vartheta + \sin \vartheta \cos \varphi D_\xi\varphi = 0 \qquad (17b)$$

In direction of $\vec{t}$,

$$\tan \mu \sin \vartheta \cos \varphi D_\xi\omega - \tan \mu \cos \vartheta D_r\omega + \cos \vartheta \cos \varphi D_\xi\vartheta$$

$$+ \sin \vartheta D_r\vartheta - \sin \vartheta \sin \varphi D_\xi\varphi = 0 \qquad (17c)$$

Condition of continuity:

$$-\cot\mu\left[\cos\vartheta D_\xi\omega + \sin\vartheta\left(\cos\varphi D_r\omega + \frac{1}{r}\sin\varphi D_\psi\omega\right)\right] - \sin\vartheta D_\xi\vartheta + \cos\vartheta\left(\cos\varphi D_r\vartheta\right.$$

$$\left.+\frac{1}{r}\sin\varphi D_\psi\vartheta\right) - \sin\vartheta\left(\sin\varphi D_r\varphi - \frac{1}{r}\cos\varphi D_\psi\varphi\right) + \frac{1}{r}\sin\vartheta\cos\varphi x_\xi = 0 \tag{17d}$$

Condition of the characteristic surface:

$$\left(\cos\vartheta - \sin\vartheta\cos\varphi x_r - \sin\vartheta\sin\varphi\frac{1}{r}x_\psi\right)^2 = \sin^2\mu\left(1 + x_r{}^2 + \frac{1}{r^2}x_\psi{}^2\right) \tag{17e}$$

These five equations are the starting point for a perturbation calculation.

Calculation of Perturbation

The system of nonlinear partial differential equations (eqs. (17)) is solved by using a perturbation method. The following assumptions are made for the unknown functions:

$$\left.\begin{aligned} \omega &= \omega^{(0)} + \omega^{(1)} + \omega^{(2)} + \ldots \\ x &= x^{(0)} + x^{(1)} + x^{(2)} + \ldots \\ \vartheta &= \vartheta^{(0)} + \vartheta^{(1)} + \vartheta^{(2)} + \ldots \\ \varphi &= \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \ldots \end{aligned}\right\} \tag{18}$$

The superscript in parentheses denotes the order of magnitude. It is assumed that for large distances r the zero-order term is greater than the first-order term, the first-order term is greater than the second-order term, and so forth. These series do not contain any assumption about the value of the orders of magnitude. The terms in the perturbation series can be calculated without making any such assumptions.

Some prior knowledge about the far flow field to the zero order can be used to shorten the calculations. For large distances r,

$$
\left.\begin{array}{lll}
\omega^{(0)} = \text{Constant} & & \\
\vartheta^{(0)} = 0 & & \\
x_{\psi}{}^{(0)} = 0 & x_{r}{}^{(0)} = \text{Constant} & x_{\xi}{}^{(0)} = \text{Constant} \\
\partial_r \sim \frac{1}{r} & \partial_\xi \sim \partial_\psi & \\
\partial_r \ll \partial_\xi & \partial_r \ll \partial_\psi &
\end{array}\right\} \tag{19}
$$

but $x_r \sim x_\psi \sim x_\xi$. The assumption $\vartheta^{(0)} = 0$ implies that the abscissa $\vec{e}^1 = \text{grad } x^1$ is in the direction of the undisturbed flow.

It is advantageous to develop each of the terms in the perturbation series in a Fourier series in ψ. Convenient forms for these developments are

$$
B^{(j)}(\xi,r,\psi) = \sum_{m=-\infty}^{\infty} B_m^{(j)}(\xi,r)e^{im\psi} \qquad (j \geqq 1) \tag{20a}
$$

where B stands for ω,μ,x,ϑ, and so forth, and

$$
\varphi^{(j)}(\xi,r,\psi) = \sum_{m=-\infty}^{\infty} i\varphi_m^{(j)}(\xi,r)e^{im\psi} \qquad (j \geqq 1) \tag{20b}
$$

Naturally, the system of equations developed in Fourier series must also be satisfied for each Fourier number m. By developing each of the five scalar differential equations (eqs. (17)) into Fourier series and suitably combining equations, one gets equations for the different orders up to the second order:

$$
\omega_{m,\xi}^{(1)} + \vartheta_{m,\xi}^{(1)} = 0 \tag{21a}
$$

$$
m\left(\omega_m^{(1)} + \vartheta_m^{(1)}\right) = 0 \tag{21b}
$$

$$
-\omega_{m,r}^{(1)} + \vartheta_{m,r}^{(1)} + \frac{1}{r}\vartheta_m^{(1)} = 0 \tag{21c}
$$

$$
\sin^2\mu^{(0)}x_{m,r}^{(1)} = -\left(\vartheta_m^{(1)} + \mu_m^{(1)}\right) \tag{21d}
$$

$$r\left(\varphi^{(1)}_{m-\nu}\vartheta^{(1)}_{\nu}\right)_{,\xi} = -m \tan \mu^{(0)}\vartheta^{(1)}_{m} \tag{21e}$$

$$\omega^{(2)}_{m,\xi} + \vartheta^{(2)}_{m,\xi} = \tan \mu^{(0)}\omega^{(1)}_{m,r} \tag{21f}$$

$$m\left(\omega^{(2)}_{m} + \vartheta^{(2)}_{m}\right) = \varphi^{(1)}_{m-\nu}\vartheta^{(1)}_{\nu} + r\left(\varphi^{(1)}_{m-\nu}\vartheta^{(1)}_{\nu}\right)_{,r} \tag{21g}$$

$$\omega^{(2)}_{m,r} + \vartheta^{(2)}_{m,r} + \frac{1}{r}\vartheta^{(2)}_{m} = \frac{1}{r}\cot \mu^{(0)}\vartheta^{(1)}_{m-\nu}\vartheta^{(1)}_{\nu} - \frac{m}{r}\varphi^{(1)}_{m-\nu}\vartheta^{(1)}_{\nu} \tag{21h}$$

$$\sin^2\mu^{(0)}x^{(2)}_{m,r} = -\left(\vartheta^{(2)}_{m} + \mu^{(2)}_{m}\right) + \sin^3\mu^{(0)}\cos \mu^{(0)}x^{(1)}_{m-\nu,r}x^{(1)}_{\nu,r} \tag{21i}$$

In these equations, the Fourier number m takes integral values in the range $-\infty \leqq m \leqq \infty$, and the summation convention is indicated by the Greek letter ν, with $-\infty \leqq \nu \leqq \infty$. In fact, for axisymmetrical body shapes one can restrict ν to $0 \leqq \nu \leqq 2$. Within this restriction the desired functions are sufficiently approximated.

Zero-, First-, and Second-Order Perturbation Solutions

In deriving the system of equations (21), the results of zero-order calculation have already been used; that is,

$$x^{(0)}_{r} = \cot \mu^{(0)}$$

and

$$\varphi^{(0)} = 0$$

Because of the special structure of equations (21), one cannot obtain any information about $x^{(0)}_{\xi}$. In linearized two-dimensional flow the characteristics are straight lines parallel to each other and obey the law

$$x = \xi + r \cot \mu$$

From this equation one estimates that in three-dimensional flow

$$x^{(0)}_{\xi} = 1 \tag{22}$$

By using the relationship between the Mach angle μ and the Prandtl-Meyer angle ω in homoenergetic flow, one gets

$$\omega = \int_{w} \cot \mu \frac{dw}{w} = \int_{M_*} \cot \mu \frac{dM_*}{M_*} = \mu + (n + 1)\tan^{-1}\left(\frac{\cot \mu}{\sqrt{n + 1}}\right) - \frac{\pi}{2} \tag{23}$$

From a Taylor series for μ in powers of ω, one obtains the following expression for μ, which is correct to second order:

$$\mu = \mu^{(0)} + c_1\left(\omega^{(1)} + \omega^{(2)}\right) + \frac{1}{2} c_2\omega^{(1)^2} \tag{24}$$

Here c_1 and c_2 are constants defined by

$$c_1 = 1 - \frac{n + 1}{n} \frac{1}{\cos^2\mu^{(0)}}$$

and

$$c_2 = -2 \frac{n + 1}{n} \frac{\sin \mu^{(0)}}{\cos^3\mu^{(0)}}$$

With the use of equation (24), this system (eqs. (21)) can be integrated in a systematic step-by-step procedure. The details of the integration are given in reference 7.

The first- and second-order solutions are as follows:

$$\left.\begin{aligned}
\vartheta_m^{(1)} &= G_m^{(1)}(\xi)r^{-1/2} \\
\omega_m^{(1)} &= -G_m^{(1)}(\xi)r^{-1/2} \\
\mu_m^{(1)} &= -c_1G_m^{(1)}(\xi)r^{-1/2} \\
x_m^{(1)}\sin^2\mu^{(0)} &= -2\left(1 - c_1\right)G_m^{(1)}(\xi)r^{1/2} \\
\varphi_{m-\nu}^{(1)}\vartheta_\nu^{(1)} &= -m \tan \mu^{(0)}r^{-3/2} \int_0^\xi G_m^{(1)}(\xi)d\xi
\end{aligned}\right\} \tag{25a}$$

$$
\left.
\begin{aligned}
\vartheta_m^{(2)} &= -\cot \mu^{(0)} G_{m-\nu}^{(1)}(\xi) G_\nu^{(1)}(\xi) r^{-1} \\
&\quad + \left(\frac{3}{8} - \frac{m^2}{2}\right) \tan \mu^{(0)} r^{-3/2} \int_0^\xi G_m^{(1)}(\xi) d\xi + G_m^{(2)}(\xi) r^{-1/2} \\
\omega_m^{(2)} &= \cot \mu^{(0)} G_{m-\nu}^{(1)}(\xi) G_\nu^{(1)}(\xi) r^{-1} \\
&\quad + \left(\frac{1}{8} + \frac{m^2}{2}\right) \tan \mu^{(0)} r^{-3/2} \int_0^\xi G_m^{(1)}(\xi) d\xi - G_m^{(2)}(\xi) r^{-1/2} \\
\mu_m^{(2)} &= c_1 \cot \mu^{(0)} G_{m-\nu}^{(1)}(\xi) G_\nu^{(1)}(\xi) r^{-1} + \frac{1}{2} c_2 \left[G_m^{(1)}(\xi)\right]^2 r^{-1} \\
&\quad + c_1 \left(\frac{1}{8} + \frac{m^2}{2}\right) \tan \mu^{(0)} r^{-3/2} \int_0^\xi G_m^{(1)}(\xi) d\xi - c_1 G_m^{(2)}(\xi) r^{-1/2} \\
x_m^{(2)} \sin^2 \mu^{(0)} &= \left\{(1 - c_1)(2 + c_1) \cot \mu^{(0)} G_{m-\nu}^{(1)}(\xi) G_\nu^{(1)}(\xi) \right. \\
&\quad \left. - \frac{1}{2} c_2 \left[G_m^{(2)}(\xi)\right]^2 \right\} \ln r - 2(1 - c_1) G_m^{(2)}(\xi) r^{1/2} \\
&\quad + \tan \mu^{(0)} \int_0^\xi G_m^{(1)}(\xi) d\xi \left[-(1 - c_1) m^2 + \frac{1}{4}(3 + c_1)\right] r^{-1/2}
\end{aligned}
\right\} \qquad (25b)
$$

The terms $G_m^{(j)}(\xi)$ are functions of integration with respect to r which are to be calculated from boundary conditions.

Coordinate of Shock Waves

In order to calculate the coordinates of the shock wave in the space Y^3, Pfriem's formula is extended to this problem. Pfriem's formula of plane steady flow states that for weak shocks the shock wave bisects the angle between the characteristics before and behind the shock (ref. 7). One can extend the formula to the space Y^3 by considering not the shock wave but the normals upon the shock surfaces.

The following definitions are used:

$\vec{n}_\infty$ normal to the generatrix of the Monge cone in the undisturbed flow

$\vec{n}_\xi$ normal to the generatrix of the Monge cone in the disturbed flow

$\vec{n}_s$ normal to the shock surface

These vectors are shown in figure 4.

The extension of Pfriem's formula to space Y^3 is given by

$$\vec{n}_\infty \cdot \vec{n}_s = \vec{n}_\xi \cdot \vec{n}_s \tag{26}$$

Analogously to equation (3), set

$$\left.\begin{aligned}\vec{n}_\infty &= \kappa_1 \text{ grad } \xi_\infty \\ \vec{n}_\xi &= \kappa_2 \text{ grad } \xi \\ \vec{n}_s &= \kappa_3 \text{ grad } \xi_s\end{aligned}\right\} \tag{27}$$

with κ_1, κ_2, and κ_3 as factors of proportionality chosen so that $\vec{n}_\infty$, $\vec{n}_\xi$, and $\vec{n}_s$ are unit vectors.

Combining equations (26) and (27) yields

$$\sin^2\mu^{(0)}\left(1 + \cot\,\mu^{(0)}x_{s,r}\right)^2\left(1 + x_r{}^2 + \frac{1}{r^2}\,x_\psi{}^2\right) = \left(1 + x_r x_{s,r} + \frac{1}{r^2}\,x_\psi x_{s,\psi}\right)^2 \tag{28}$$

Equation (28) must be developed in the same way as equations (17) to obtain a set analogous to equations (21). Up to the second order, this set is

$$\left.\begin{aligned}x_{s,r}{}^{(0)} &= \cot\,\mu^{(0)} \\ x_{s,r}{}^{(1)} &= \frac{1}{2}\,x_r{}^{(1)} = -\frac{1}{2\sin^2\mu^{(0)}}\left(\vartheta^{(1)} + \mu^{(1)}\right) \\ x_{s,r}{}^{(2)}\sin^2\mu^{(0)} &= -\frac{1}{2}\left(\vartheta^{(2)} + \mu^{(2)}\right) + \sin^3\mu^{(0)}\cos\,\mu^{(0)}\left(x_{s,r}{}^{(1)}\right)^2\end{aligned}\right\} \tag{29}$$

These equations can be integrated to give

$$
\left.
\begin{aligned}
&x_s^{(0)} = r \cot \mu^{(0)} + \xi \\
&x_{s_m}^{(1)} \sin \mu^{(0)} = -\left(1 - c_1\right) G_m^{(1)}(\xi) r^{1/2} \\
&x_{s_0}^{(2)} \sin^2 \mu^{(0)} = -\left(1 - c_1\right) G_0^{(2)}(\xi) r^{1/2} \\
&\qquad + \left[\left(\frac{1 - c_1}{4}\right)\left(3 + c_1\right)\cot \mu^{(0)} - \frac{c^2}{4}\right]\left[G_0^{(1)}(\xi)\right]^2 \ln r \\
&\qquad + \frac{3 + c_1}{8} \tan \mu^{(0)} r^{-1/2} \int_0^{\xi} G_0^{(1)}(\xi) d\xi
\end{aligned}
\right\} \qquad (30)
$$

For general three-dimensional flow, the solutions are valid up to first order; for axisymmetrical flow, up to second order.

Satisfying the Boundary Condition

In equations (25) and (30) the functions of integration $G_m^{(j)}(\xi)$ were introduced. The functions $G_m^{(1)}(\xi)$ and $G_m^{(2)}(\xi)$ must be calculated from the boundary conditions. Unfortunately, the boundary conditions are formulated at the body itself though system (21) has been integrated for large distances r away from the body. In this paper, the boundary condition will be obtained from slender-body theory, a linearization of the singularity method. Use of this linearization is another approximation. But slender-body theory has the advantage of short, compact solutions. As system (21) has been integrated for large distances r, the solutions cannot be continued to the body itself to satisfy the boundary condition there. One must satisfy the boundary condition at some distance R away from the body. For this purpose, the following procedure is used.

It is assumed that the disturbances sent out by the body run along straight characteristics (zero-order theory). At the distance R the straight characteristics are matched with those characteristics which result from the exact theory and which are valid for large distances. A discussion of the matching distance R is presented subsequently in connection with an example calculation.

From a physical standpoint, a proper way to calculate the functions of integration is by considering the angle of inclination of the streamline. The inclination may generally be expressed by

$$\tan \vartheta = \frac{v}{u_\infty + u}$$

Here v denotes a vertical velocity component and $u_\infty + u$, a horizontal velocity component. For small disturbances this equation can be approximated as

$$\vartheta = \frac{v}{u_\infty} = \beta \frac{\partial \phi}{\beta\ \partial r} = \beta \frac{\partial \phi}{\partial \eta} = \beta \phi_\eta \tag{31}$$

where $\beta = \sqrt{M^2 - 1}$ and $\eta = \beta r$. The boundary condition is satisfied by using the equation.

Developed in perturbation and Fourier series, equation (31) can be rewritten as

$$\vartheta_0^{(1)} + \vartheta_0^{(2)} + \left(\vartheta_1^{(1)} + \vartheta_1^{(2)}\right) 2 \cos \psi + \left(\vartheta_2^{(1)} + \vartheta_2^{(2)}\right) 2 \cos 2\psi$$

$$= \left[\phi_{\eta_0}^{(1)} + \phi_{\eta_0}^{(2)} + \left(\phi_{\eta_1}^{(1)} + \phi_{\eta_1}^{(2)}\right) 2 \cos \psi + \left(\phi_{\eta_2}^{(1)} + \phi_{\eta_2}^{(2)}\right) 2 \cos 2\psi\right]\beta \tag{32}$$

By comparing equation (32) with equations (25), the function of integration can be expressed in terms of ϕ_η, which itself can be determined by slender-body theory. The comparison gives

$$\left.\begin{aligned}
G_0^{(1)}(\xi) &= \sqrt{R}\beta\phi_{\eta_0}^{(1)} \\
G_0^{(2)}(\xi) &= \sqrt{R}\beta\phi_{\eta_0}^{(2)} + \cot \mu^{(0)} G_{-\nu}^{(1)}(\xi) G_\nu^{(1)}(\xi) R^{-1/2} - \frac{3}{8} \tan \mu^{(0)} R^{-1} \int_0^\xi G_0^{(1)}(\xi) d\xi \\
G_1^{(1)}(\xi) &= \sqrt{R}\beta\phi_{\eta_1}^{(1)} \\
G_1^{(2)}(\xi) &= \sqrt{R}\beta\phi_{\eta_1}^{(2)} + \cot \mu^{(0)} G_{1-\nu}^{(1)}(\xi) G_\nu^{(1)}(\xi) R^{-1/2} + \frac{1}{8} \tan \mu^{(0)} R^{-1} \int_0^\xi G_1^{(1)}(\xi) d\xi \\
G_2^{(1)}(\xi) &= \sqrt{R}\beta\phi_{\eta_2}^{(1)} \\
G_2^{(2)}(\xi) &= \sqrt{R}\beta\phi_{\eta_2}^{(2)} + \cot \mu^{(0)} G_{2-\nu}^{(1)}(\xi) G_\nu^{(1)}(\xi) R^{-1/2} + \frac{13}{8} \tan \mu^{(0)} R^{-1} \int_0^\xi G_2^{(1)}(\xi) d\xi
\end{aligned}\right\} \tag{33}$$

In order to determine the quantity ϕ_η, a modified equation given by von Kármán and Moore (ref. 8) is considered:

$$\phi = \sum_{m=-\infty}^{\infty} \int_{z=0}^{x-\eta} f_m(z) \frac{\sinh m\lambda}{m} dz e^{im\psi}$$

Here $f_m(z)$ denotes source distribution functions (which are given subsequently) and

$$\sinh \lambda = \sqrt{\cosh^2\lambda - 1} = \sqrt{\frac{(x - z)^2}{\eta^2} - 1}$$

The derivative with respect to $\eta = \beta r$ gives

$$\phi_\eta = - \sum_{m=-\infty}^{\infty} \int_{z=0}^{x-\eta} f_m(z) \frac{\cosh m\lambda \cosh \lambda}{\eta \sinh \lambda} dz e^{im\psi} \tag{34}$$

Developing the integrals in equation (34) for slender bodies, or small η, gives the functions f_m. Since only symmetrical bodies are considered, one can restrict m to $0 \leqq m \leqq 2$ and obtain

$$\begin{aligned}\phi_\eta = &-\int_0^{x-\eta} f_0 \frac{\cosh \lambda}{\eta \sinh \lambda} dz - \int_0^{x-\eta} f_1 \frac{\cosh^2\lambda}{\eta \sinh \lambda} dz\left(e^{i\psi} + e^{-i\psi}\right) \\ &-\int_0^{x-\eta} f_2 \frac{\cosh 2\lambda \cosh \lambda}{\eta \sinh \lambda} dz\left(e^{2i\psi} + e^{-2i\psi}\right)\end{aligned} \tag{35}$$

The functions $f_m(z)$ which follow from slender-body theory are

$$\left.\begin{aligned}&\int_0^{x-\eta} f_0(z)dz = f_0^{(-1)}(x - \eta) = -K_0(x - \eta)\frac{dK_0}{dx} = -K_0(x - \eta)K_0'(x - \eta) \\ &f_1^{(-2)}(x - \eta) = -\frac{\eta^2}{\beta} K_1'(x - \eta) - \frac{\alpha\eta^2}{2\beta} \\ &f_2^{(-3)}(x - \eta) = -\frac{\eta^3}{4\beta} K_2'(x - \eta)\end{aligned}\right\} \tag{36}$$

The angle of attack of the body α is measured counterclockwise, as shown in figure 5. The radius of the body is denoted as $K(x,\psi)$ and can be developed in a Fourier series as $K(x,\psi) = \sum_{m=-\infty}^{\infty} K_m(x)e^{im\psi}$. Since one supposes equations (36) to be introduced in equation (35) and, after that, equation (35) developed in first and second order, one is able to calculate the functions of integration (eqs. (33)). This procedure shall now be demonstrated for a body of revolution.

The Body of Revolution Flying at an Angle of Attack α

For a body of revolution the radius is simply

$$K(x,\psi) = K_0(x)$$

Therefore, the source distribution functions (eqs. (36)) are

$$\left.\begin{aligned} &f_0^{(-1)}(x - \eta) = -K_0(x - \eta)K_0'(x - \eta) \\ &\text{or} \\ &f_0(x - \eta) = -\frac{1}{2\pi} A''(x - \eta) \end{aligned}\right\} \tag{37}$$

where $A = \pi K^2$ denotes the cross-sectional area of the body and

$$\left.\begin{aligned} &f_1^{(-2)}(x - \eta) = -\frac{\alpha\eta^2}{2\beta} \\ &\text{or} \\ &f_1(x - \eta) = -\frac{1}{2\pi}\alpha\beta A''(x - \eta) \end{aligned}\right\} \tag{38}$$

The source distribution functions with higher Fourier numbers are all equal to zero. Thus, for a lifting body of revolution, only the Fourier numbers $m = 0$ and $m = 1$ have to be taken into account. By putting these numbers in the set of solutions given by equations (25) and referring to equation (20a), it can be seen that

$$\vartheta^{(j)} = \vartheta_0^{(j)} + \vartheta_1^{(j)}e^{i\psi} + \vartheta_{-1}^{(j)}e^{-i\psi}$$

or because of the symmetry at $\psi = 0$,

$$\vartheta^{(j)} = \vartheta_0^{(j)} + 2\vartheta_1^{(j)} \cos\psi$$

The solutions can then be determined.

The solutions for ϑ and x are written as

$$\left.\begin{aligned}
\vartheta^{(1)} &= \vartheta_0^{(1)} + \vartheta_1^{(1)} 2\cos\psi = \frac{1}{\sqrt{r}}\left(G_0^{(1)} + 2\cos\psi G_1^{(1)}\right) \\
x^{(1)} &= -\frac{2\left(1 - c_1\right)}{\sin^2\mu^{(0)}}\sqrt{r}\left(G_0^{(1)} + 2\cos\psi G_1^{(1)}\right)
\end{aligned}\right\} \tag{39a}$$

and

$$\left.\begin{aligned}
\vartheta^{(2)} &= \vartheta_0^{(2)} + 2\vartheta_1^{(2)} \cos\psi \\
&= -\cot\mu^{(0)} r^{-1}\left[\left(G_0^{(1)}\right)^2 + 2\left(G_1^{(1)}\right)^2 + 4\cos\psi G_0^{(1)} G_1^{(1)}\right] + r^{-1/2}\left(G_0^{(2)} + 2\cos\psi G_1^{(2)}\right) \\
&\quad + \frac{1}{8}\tan\mu^{(0)} r^{-3/2}\left[3\int_0^{\xi} G_0^{(1)}(\xi)d\xi - 2\cos\psi\int_0^{\xi} G_1^{(1)}(\xi)d\xi\right] \\
\sin^2\mu^{(0)} x^{(2)} &= \Bigg\{\left(1 - c_1\right)\left(2 + c_1\right)\cot\mu^{(0)}\left[\left(G_0^{(1)}\right)^2 + 2\left(G_1^{(1)}\right)^2 + 4\cos\psi G_0^{(1)} G_1^{(1)}\right] \\
&\quad - \frac{1}{2} c_2\left[\left(G_0^{(1)}\right)^2 + 2\cos\psi\left(G_1^{(1)}\right)^2\right]\Bigg\}\ln r \\
&\quad - 2\left(1 - c_1\right) r^{1/2}\left(G_0^{(2)} + 2\cos\psi G_1^{(2)}\right) \\
&\quad + \tan\mu^{(0)}\frac{\left(3 + c_1\right)}{4} r^{-1/2}\left[\int_0^{\xi} G_0^{(1)}(\xi)d\xi + \int_0^{\xi} G_1^{(1)}(\xi)d\xi\right] \\
&\quad - \tan\mu^{(0)}\left(1 - c_1\right) r^{-1/2}\int_0^{\xi} G_1^{(1)}(\xi)d\xi
\end{aligned}\right\} \tag{39b}$$

Calculation of the Functions of Integration

For a lifting body of revolution, the functions of integration (eqs. (33)) can be written as

$$G_0^{(1)} = \beta\sqrt{R}\phi_{\eta_0}^{(1)}$$

$$G_1^{(1)} = \beta\sqrt{R}\phi_{\eta_1}^{(1)}$$

$$G_0^{(2)} = \beta\sqrt{R}\phi_{\eta_0}^{(2)} + \cot \mu^{(0)}\left[\left(G_0^{(1)}\right)^2 + 2\left(G_1^{(1)}\right)^2\right]R^{-1/2} - \frac{3}{8}\tan \mu^{(0)}R^{-1}\int_0^{\xi} G_0^{(1)}(\xi)d\xi$$

and

$$G_1^{(2)} = \beta\sqrt{R}\phi_{\eta_1}^{(2)} + 2 \cot \mu^{(0)}G_0^{(1)}G_1^{(1)}R^{-1/2} + \frac{1}{8}\tan \mu^{(0)}R^{-1}\int_0^{\xi} G_1^{(1)}(\xi)d\xi$$

Here, the quantities $\phi_{\eta_m}^{(j)}$ are to be determined. This determination can be made, as mentioned previously, by using the slender-body theory (eq. (34)). By using the derived source distribution functions for a lifting body of revolution (eqs. (37) and (38)), equation (34) can be written as

$$\left.\begin{aligned}
\phi_{\eta_0} &= \frac{1}{2\pi\eta}\int_0^{x-\eta} \frac{(x - z)A''(z)}{\sqrt{(x - z)^2 - \eta^2}}\,dz \\
\phi_{\eta_1} &= \frac{\alpha\beta}{2\pi\eta^2}\int_0^{x-\eta} \frac{(x - z)^2A''(z)dz}{\sqrt{(x - z)^2 - \eta^2}}
\end{aligned}\right\} \qquad (40)$$

or, because $\xi + \eta = x$, as

$$\left.\begin{aligned}
\phi_{\eta_0} &= \frac{1}{2\pi\eta}\int_0^{\xi} \frac{(\xi + \eta - z)A''(z)dz}{\sqrt{\xi - z}\sqrt{\xi + 2\eta - z}} \\
\phi_{\eta_1} &= \frac{\alpha\beta}{2\pi\eta^2}\int_0^{\xi} \frac{(\xi + \eta - z)^2A''(z)dz}{\sqrt{\xi - z}\sqrt{\xi + 2\eta - z}}
\end{aligned}\right\} \qquad (41)$$

In calculating the functions of integration η means here $\eta = \beta R$.

By integration by parts the singularity at $z = \xi$ can be removed. The result is

$$\phi_{\eta_0} = \frac{1}{2\pi\eta}\left[-2\sqrt{\xi - z}\,\frac{(\xi + \eta - z)}{\sqrt{\xi + 2\eta - z}}\,A''(z)\Big|_0^{\xi} + 2\int_0^{\xi}\sqrt{\xi - z}\,\frac{(\xi + \eta - z)}{\sqrt{\xi + 2\eta - z}}\,A'''(z)dz\right.$$

$$\left. - \int_0^{\xi}\sqrt{\xi - z}\,\frac{(\xi + 3\eta - z)}{\sqrt{(\xi + 2\eta - z)^3}}\,A''(z)dz\right]$$

After the development of the integrand in power series, the integration is possible. As a result of these operations, the first term of ϕ_{η_0} vanishes. Since the values of η considered herein are large ($\eta > 10$) and the values of ξ are small ($\xi = 0$ to 1), it can be seen that the magnitude of the second term of ϕ_{η_0} is about η times larger than the third term of ϕ_{η_0}. Therefore, the second term is of the first order and the third term is of the second order. One gets for pointed bodies ($A(0) = A'(0) = 0$):

$$\left.\begin{aligned} \phi_{\eta_0}^{(1)} &= \frac{1}{2\pi\eta}\left[\frac{\sqrt{\xi}(\xi + \eta)}{\sqrt{\xi + 2\eta}}\left(\frac{A'}{\xi} + \frac{A1}{2\xi^2} + \frac{3A2}{8\xi^3} + \frac{5A3}{16\xi^4} + \ldots\right)\right] \\ \phi_{\eta_0}^{(2)} &= \frac{1}{2\pi\eta}\left[\frac{\sqrt{\xi}(\xi + 3\eta)}{\sqrt{(\xi + 2\eta)^3}}\left(A' - \frac{A1}{2\xi} - \frac{A2}{8\xi^2} - \frac{A3}{16\xi^3} + \ldots\right)\right] \end{aligned}\right\} \qquad (42)$$

Here and in equations (43) and (44),

$$A' = \frac{dA}{d\xi} = 2\pi K\,\frac{dK}{d\xi}$$

$$A1 = A'\xi - A$$

$$A2 = A'\xi^2 - 2A\xi + 2\int_0^{\xi} A(z)dz$$

and

$$A3 = A'\xi^3 - 3A\xi^2 + 6\int_0^{\xi} A(z)z\,dz$$

Analogously, the term ϕ_{η_1} is calculated to be

$$\phi_{\eta_1} = \frac{\alpha\beta}{2\pi\eta^2}\int_0^{\xi} \frac{(\xi + \eta - z)^2 A''(z)dz}{\sqrt{\xi - z}\sqrt{\xi + 2\eta - z}}$$

Hence,

$$\left.\begin{aligned}
\phi_{\eta_1}^{(1)} &= \frac{\alpha\beta}{2\pi\eta^2}\,\frac{\sqrt{\xi}(\xi + \eta)^2}{\sqrt{\xi + 2\eta}}\left(\frac{A'}{\xi} + \frac{A1}{2\xi^2} + \frac{3A2}{8\xi^3} + \frac{5A3}{16\xi^4} + \ldots\right) \\
\phi_{\eta_1}^{(2)} &= \frac{\alpha\beta}{2\pi\eta^2}\,\frac{\sqrt{\xi}\left(3\xi^2 + 10\eta\xi + 7\eta^2\right)}{2\sqrt{(\xi + 2\eta)^3}}\left(-A' + \frac{A1}{2\xi} + \frac{A2}{8\xi^2} + \frac{A3}{16\xi^3} + \ldots\right)
\end{aligned}\right\} \qquad (43)$$

The functions of integration are determined as

$$\left.\begin{aligned}
G_0^{(1)} &= \frac{\sqrt{\xi}(\xi + \eta)}{2\pi\sqrt{R}\sqrt{\xi + 2\eta}}\left(\frac{A'}{\xi} + \frac{A1}{2\xi^2} + \frac{3A2}{8\xi^3} + \frac{5A3}{16\xi^4} + \ldots\right) \\
G_1^{(1)} &= \frac{\alpha\sqrt{\xi}(\xi + \eta)^2}{2\pi R^{3/2}\sqrt{\xi + 2\eta}}\left(\frac{A'}{\xi} + \frac{A1}{2\xi^2} + \frac{3A2}{8\xi^3} + \frac{5A3}{16\xi^4} + \ldots\right)
\end{aligned}\right\} \qquad (44a)$$

and

$$\left.\begin{aligned}
G_0^{(2)} &= \frac{\sqrt{\xi}(\xi + 3\eta)}{2\pi\sqrt{R}\sqrt{(\xi + 2\eta)^3}}\left(-A' + \frac{A1}{2\xi} + \frac{A2}{8\xi^2} + \frac{A3}{16\xi^3} + \ldots\right) \\
&\quad + \cot\mu^{(0)}\left[\left(G_0^{(1)}\right)^2 + 2\left(G_1^{(1)}\right)^2\right]R^{-1/2} - \frac{3}{8}\tan\mu^{(0)}R^{-1}\int_0^{\xi} G_0^{(1)}(\xi)d\xi \\
G_1^{(2)} &= \frac{\alpha\sqrt{\xi}\left(3\xi^2 + 10\eta\xi + 7\eta^2\right)}{4\pi R^{3/2}\sqrt{(\xi + 2\eta)^3}}\left(-A' + \frac{A1}{2\xi} + \frac{A2}{8\xi^2} + \frac{A3}{16\xi^3} + \ldots\right) \\
&\quad + 2\cot\mu^{(0)}\left(G_0^{(1)}G_1^{(1)}\right)R^{-1/2} - \frac{7}{8}\tan\mu^{(0)}R^{-1}\int_0^{\xi} G_1^{(1)}(\xi)d\xi
\end{aligned}\right\} \qquad (44b)$$

The Sonic-Boom Signature

Finally the sonic-boom signature is calculated. The pressure coefficient is

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = C_p = -2\phi_x$$

where ϕ_x is the disturbed velocity in dimensionless form in the direction of $\vec{e}^1$ and where the subscript ∞ characterizes the values of the free-stream flow. The disturbed velocity ϕ_x is

$$\phi_x = \frac{w}{u_\infty} \cos \vartheta - 1$$

Hence, the pressure coefficient becomes

$$\frac{p - p_\infty}{p_\infty} = -\frac{\rho_\infty}{p_\infty} M_\infty a_\infty^2 \left(M_* \frac{a_*}{a_\infty} \cos \vartheta - M_\infty\right)$$

Using the energy equation of homoenergetic flow

$$n a_\infty^2 + u_\infty^2 = (n + 1) a_*^2$$

and the definition of speed of sound

$$a_\infty^2 = \frac{n + 2}{n} \frac{p_\infty}{\rho_\infty}$$

one gets

$$\frac{p - p_\infty}{p_\infty} = -\frac{n + 2}{n} M_\infty \left(\cos \vartheta M_* \sqrt{\frac{n + M_\infty^2}{n + 1}} - M_\infty\right) \tag{45}$$

The term M_* is given by the Prandtl-Meyer angle $\omega = \int_\mu \cot \mu \frac{dM_*}{M_*}$ and is a function of μ and ω. The integration is possible because μ can be expressed as a function of M_*. The integration gives

$$M_*^2 = \frac{(n + 1)\tan^2\left(\frac{\omega - \mu + \frac{\pi}{2}}{n + 1}\right) + 1}{\tan^2\left(\frac{\omega - \mu + \frac{\pi}{2}}{n + 1}\right) + 1}$$

where

$$\mu = \mu^{(0)} + \mu^{(1)} + \mu^{(2)} = \sin^{-1}\frac{1}{M_\infty} + c_1\left(\omega^{(1)} + \omega^{(2)}\right) + \frac{1}{2}c_2{\omega^{(1)}}^2$$

and where

$$\omega = \omega^{(0)} + \omega^{(1)} + \omega^{(2)}$$

The term $\omega^{(0)}$ is given by

$$\omega^{(0)} = \mu^{(0)} + \sqrt{n+1}\tan^{-1}\sqrt{\frac{{M_\infty}^2 - 1}{n+1}} - \frac{\pi}{2}$$

and $\omega^{(1)}$ and $\omega^{(2)}$ are given by equations (25).

In equation (45),

$$\vartheta = \vartheta^{(1)} + \vartheta^{(2)}$$

and

$$\cos\vartheta = \cos\left(\vartheta^{(1)} + \vartheta^{(2)}\right)$$

The terms $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are also given by equations (25).

Comparison With Result of Reference 4

For a body of revolution at zero angle of attack, the equation of the abscissa x is

$$x = r\cot\mu^{(0)} + \xi - 2\frac{n+1}{n}\frac{1}{\sin^2\mu^{(0)}\cos^2\mu^{(0)}}G_0^{(1)}(\xi)\sqrt{r}$$

$$- 2\frac{n+1}{n}\frac{1}{\sin^2\mu^{(0)}\cos^2\mu^{(0)}}G_0^{(2)}(\xi)\sqrt{r}$$

$$+ \frac{4n\cos^2\mu^{(0)} - n - 1}{4n\cos^3\mu^{(0)}\sin\mu^{(0)}}\int_0^\xi G_0^{(1)}(\xi)d\xi r^{-1/2}$$

$$+ \frac{2n+1}{n}\frac{n+1}{n}\frac{1}{\cos^3\mu^{(0)}\sin^3\mu^{(0)}}\left[G_0^{(1)}(\xi)\right]^2\ln r$$

By using $n = \frac{2}{\gamma - 1}$ and $\cot^2\mu^{(0)} = M_\infty^2 - 1$, the previous equation becomes

$$x = r\sqrt{M_\infty^2 - 1} + \xi - (\gamma - 1)\frac{M_\infty^4}{M_\infty^2 - 1} G_0^{(1)}(\xi)\sqrt{r} - (\gamma + 1)\frac{M_\infty^4}{M_\infty^2 - 1} G_0^{(2)}(\xi)\sqrt{r}$$

$$+ \frac{\left(M_\infty^2 - 1\right)M_\infty^2 - \frac{1}{8}(\gamma + 1)M_\infty^4}{\left(M_\infty^2 - 1\right)^{3/2}} \int_0^\xi G_0^{(1)}(\xi)d\xi r^{-1/2}$$

$$+ \frac{(\gamma + 1)(\gamma + 3)}{2} \frac{M_\infty^6}{\sqrt{\left(M_\infty^2 - 1\right)^3}}\left[G_0^{(1)}(\xi)\right]^2 \ln r$$

In comparison, Whitham's result (ref. 4, eq. (99)) is

$$x = r\sqrt{M_\infty^2 - 1} + \xi - (\gamma + 1)\frac{M_\infty^4}{\sqrt{M_\infty^2 - 1}} F(\xi)\sqrt{r}$$

$$- \frac{M_\infty^6}{\sqrt{\left(M_\infty^2 - 1\right)^3}} \frac{\gamma + 1}{4}\left[M_\infty^2(\gamma + 4) - (2\gamma + 5)\right]\left[F(\xi)\right]^2 \log r$$

Though it is impossible to equate the terms of both solutions as different methods of calculations have been used, similarities in coefficients can be seen. Above all, the comparison shows that, in general, both solutions obey the same power series. But one must recognize that in the present paper no approximations have been made in obtaining the solutions of the nonlinear differential equations; however, approximations are made in evaluating the functions of integration. The slender-body theory has been used in both papers to calculate the functions of integration. An extension of Whitham's theory for slender lifting bodies is given in reference 9.

EXAMPLE CALCULATIONS

The sonic-boom pressure signature has been calculated for different Mach numbers for two bodies of revolution. The cross-sectional areas of the two bodies are given in dimensionless form by

$$A_1 = 0.04\left(x^2 - 2x^3 + x^4\right)\pi \qquad (0 \leqq x \leqq 1)$$

and

$$A_2 = 0.01\left(4x^2 - 4x^3 + x^4\right)\pi \qquad (0 \leqq x \leqq 1)$$

The numerical evaluation is made by using equaticn (45).

In figure 6 the functions of integration $G^{(1)}(\xi)$ and $G^{(2)}(\xi)$ are drawn for the two body shapes considered. As was mentioned previously and is illustrated in this figure, the second-order integration function is much smaller than the first-order integration function; this fact provides a justification for restricting oneself to a first-order calculation. Thus, the influence of the second-order calculation has been neglected in the example calculations because of the appropriate choice of the first- and second-order integration functions (eqs. (44)).

The boundary condition was satisfied at some distances R away from the body. As shown in figure 7, the function of integration $G^{(1)}(\xi)$ is independent of the value of the matching parameter R over the range $3 \leqq R \leqq 50$. For smaller values of R, the solutions broke down as the set of differential equations was solved for large distances r, whereas for larger values of R, the boundary condition was not accurately represented by slender-body theory.

As shown in figure 8, the point of intersection of the Mach line of the undisturbed flow through the nose of the body and a parallel line to the abscissa x at the radial distance r gives the origin of the plots of the pressure signatures shown in figures 9 to 11. In figure 9 different pressure signatures for one body of revolution are drawn. The angle of attack α, the azimuth angle ψ, and the radial distance r where the pressure signature is calculated are fixed. Here, the pressure signature is a function of the distance R at which the boundary condition is satisfied. It can be seen that the different values of the matching parameter R do not influence the shape of the pressure signature. In all the sample calculations only the bow shock was calculated according to equations (30).

Figures 10 and 11 show the influence of the factor $\alpha \cos \psi$ and the radial distance r on the pressure signature. Naturally, the pressure rise increases with decreasing distance r and increasing Mach number. Furthermore, the pressure rise for a lifting body is greater at the bottom side ($\psi = 180^{\circ}$) than at the top side ($\psi = 0^{\circ}$).

In figure 12 a comparison of the results computed by the present theory and those computed by Whitham's theory (ref. 4) is shown. The agreement between these theories is rather good. However, it must be emphasized that the present theory has advantages such as simplicity of equations, handling of lifting bodies, and easy extension to more general problems as the change from a homogeneous to an inhomogeneous atmosphere.

CONCLUDING REMARKS

The exact nonlinear system of partial differential equations for supersonic flow was solved for large distances by using a perturbation method developed by Poincaré, Lighthill, and Kuo. The unknown functions were expanded in a perturbation series. The system of differential equations were derived for each order of magnitude and the unknown functions obtained by step-by-step integration. The integration introduced arbitrary functions of integration which could be used to satisfy boundary conditions at the body. The boundary conditions were obtained as in previous studies of nonlifting bodies by using slender-body theory. However, axisymmetric lifting bodies with nearly circular cross sections were considered herein. In order to calculate the flow away from the body, the boundary conditions were satisfied not at the body itself, but at some distance R away from the body. The example calculations give reasonable sonic-boom signatures for arbitrarily chosen bodies of revolution. The results are in good agreement with the previous computations for nonlifting bodies.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., July 20, 1971.

REFERENCES

1. Lansing, Donald L.: Application of Acoustic Theory to Prediction of Sonic-Boom Ground Patterns From Maneuvering Aircraft. NASA TN D-1860, 1964.

2. Hayes, Wallace D.; Haefeli, Rudolph C.; and Kulsrud, H. E.: Sonic Boom Propagation in a Stratified Atmosphere, With Computer Program. NASA CR-1299, 1969.

3. Friedman, Manfred P.; Kane, Edward J.; and Sigalla, Armand: Effects of Atmosphere and Aircraft Motion on the Location and Intensity of a Sonic Boom. AIAA J., vol. 1, no. 6, June 1963, pp. 1327-1335.

4. Whitham, G. B.: The Behaviour of Supersonic Flow Past a Body of Revolution, Far From the Axis. Proc. Roy. Soc. (London), ser. A, vol. 201, no. 1064, Mar. 7, 1950, pp. 89-109.

5. Tsien, H. S.: The Poincaré-Lighthill-Kuo Method. Vol. IV of Advances in Applied Mechanics, H. L. Dryden and Th. von Kármán, eds., Academic Press, Inc., 1956, pp. 281-349.

6. Sauer, Robert: Nichtstationäre Probleme der Gasdynamik. Springer Verlag, 1966.

7. Schorling, Michael: Berechnung von Überschallströmungen weit entfernt von schlanken Flugkörpern mit Hilfe der Charakteristikentheorie nach der P.L.K. Methode. Dr.-Ing. Diss., Rheinisch-Westfälischen Technischen Hochschule, 1969.

8. Von Kármán, Theodor; and Moore, Norton B.: Resistance of Slender Bodies Moving With Supersonic Velocities, With Special Reference to Projectiles. Trans. ASME, vol. 54, no. 23, Dec. 15, 1932, pp. 303-310.

9. Walkden, F.: The Shock Pattern of a Wing-Body Combination, Far From the Flight Path. Aeron. Quart., vol. IX, pt. 2, May 1958, pp. 164-194.

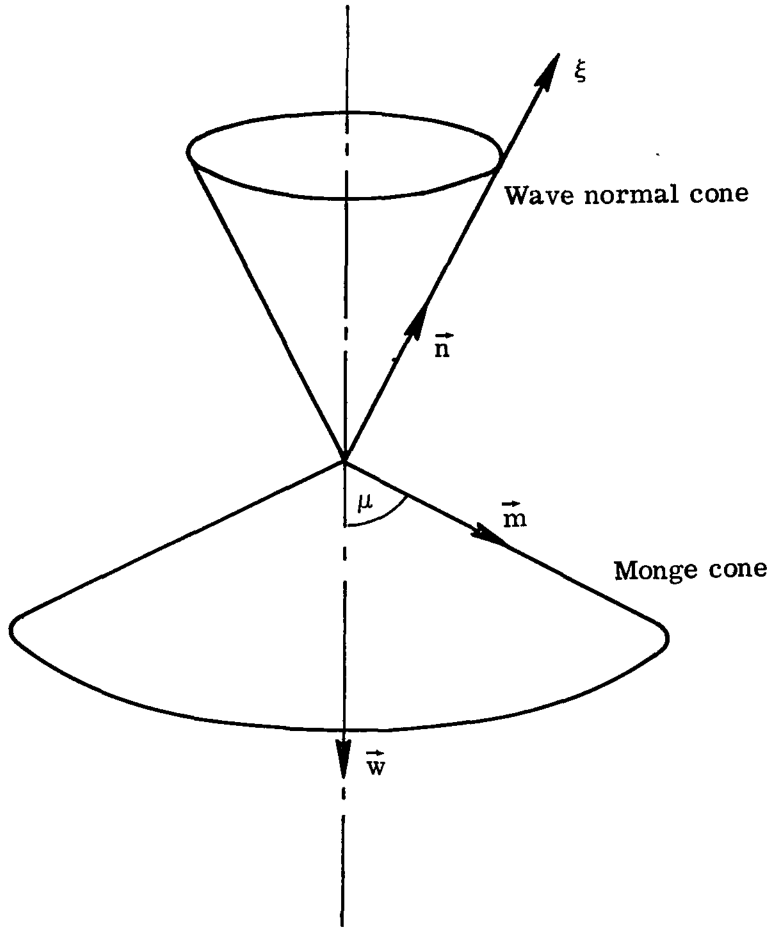


Figure 1.- Monge cone and wave normal cone.

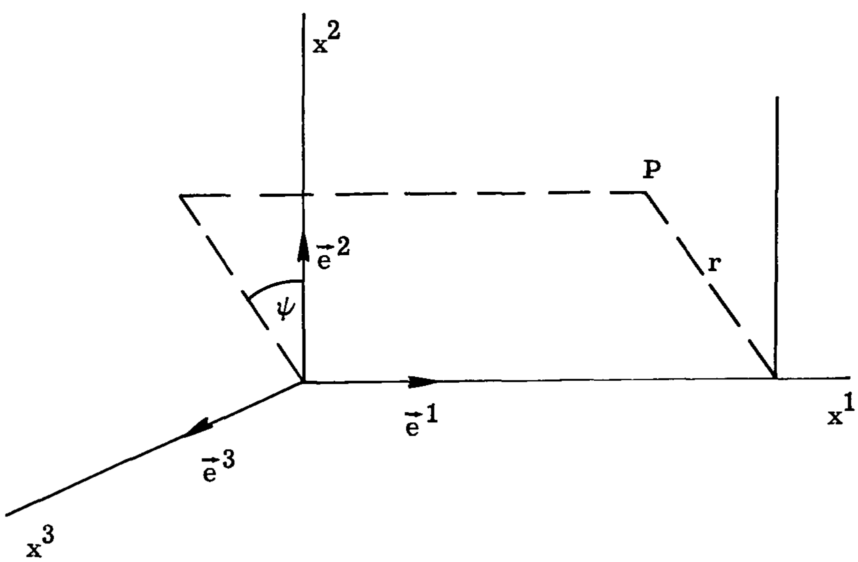


Figure 2.- The coordinates of point P.

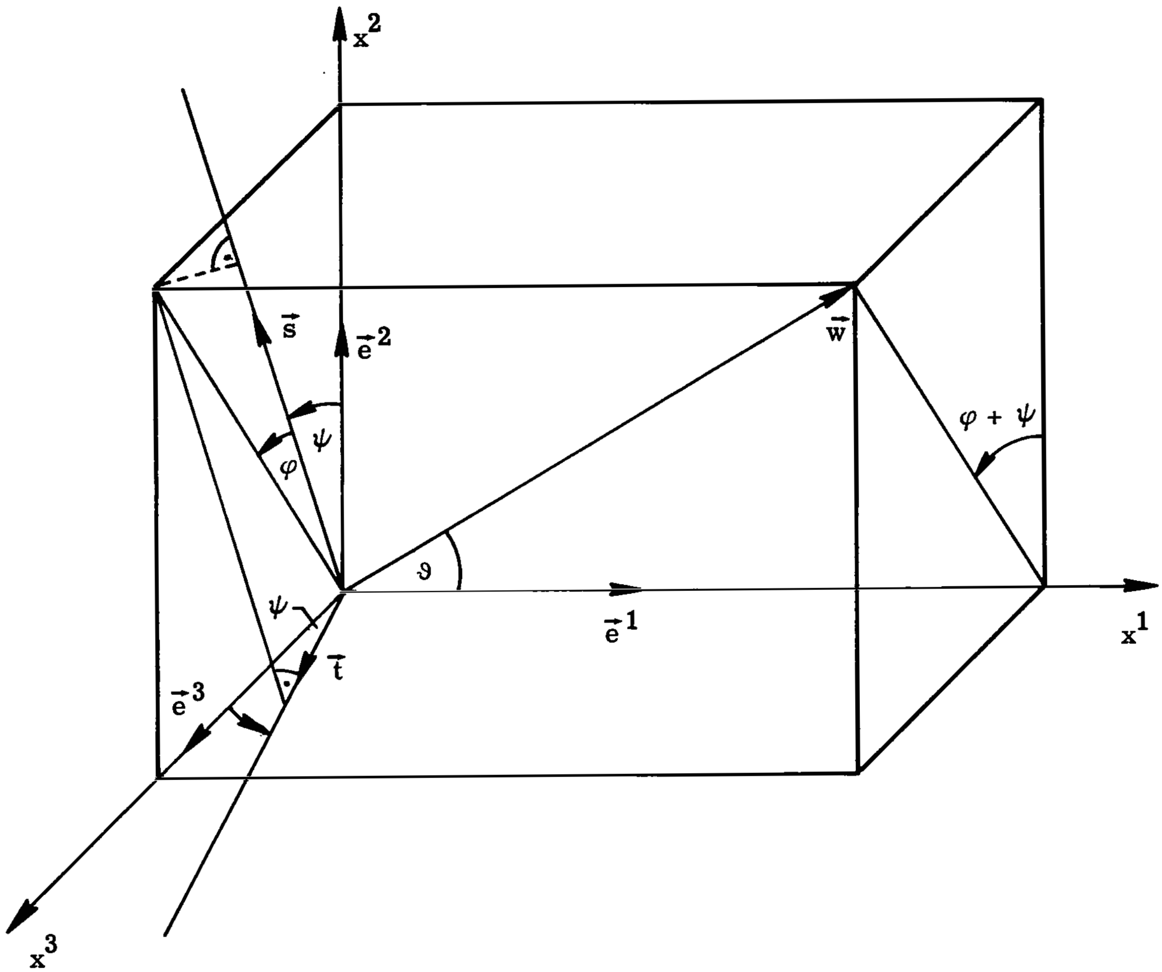


Figure 3.- The velocity vector.

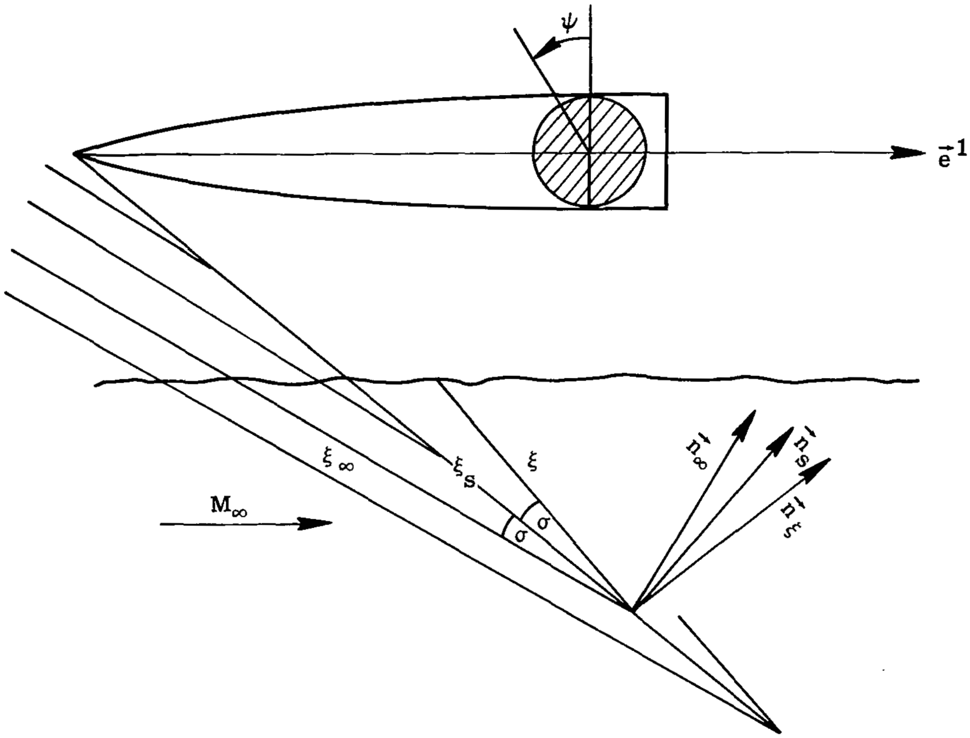


Figure 4.- Description of the weak shock (Pfriem's formula).

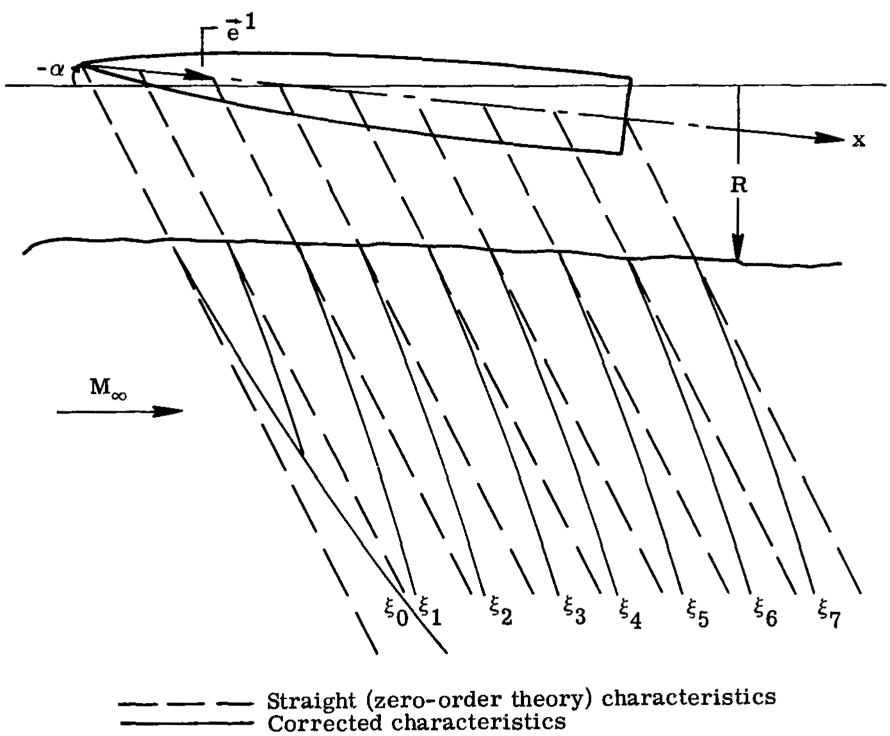
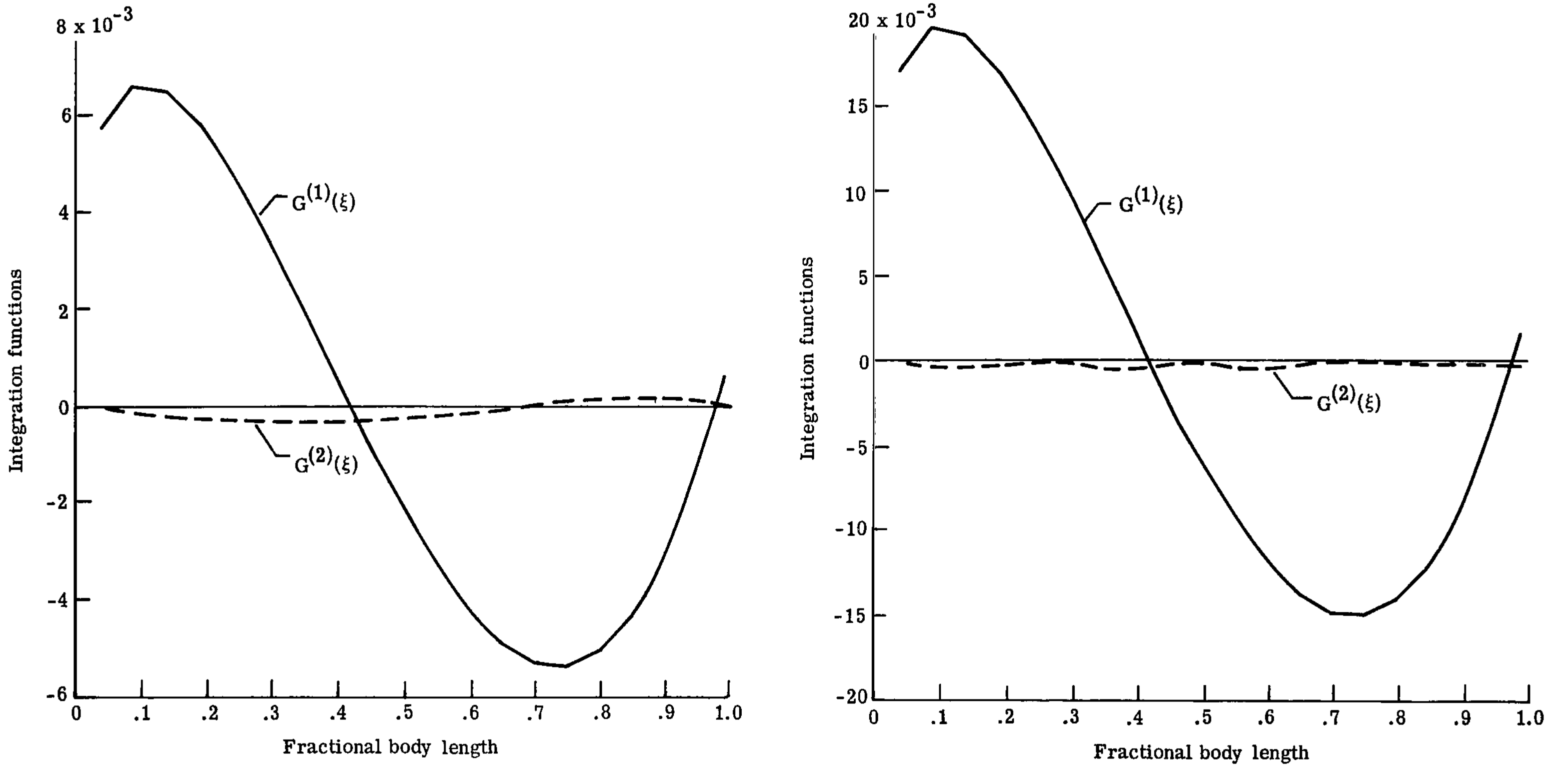


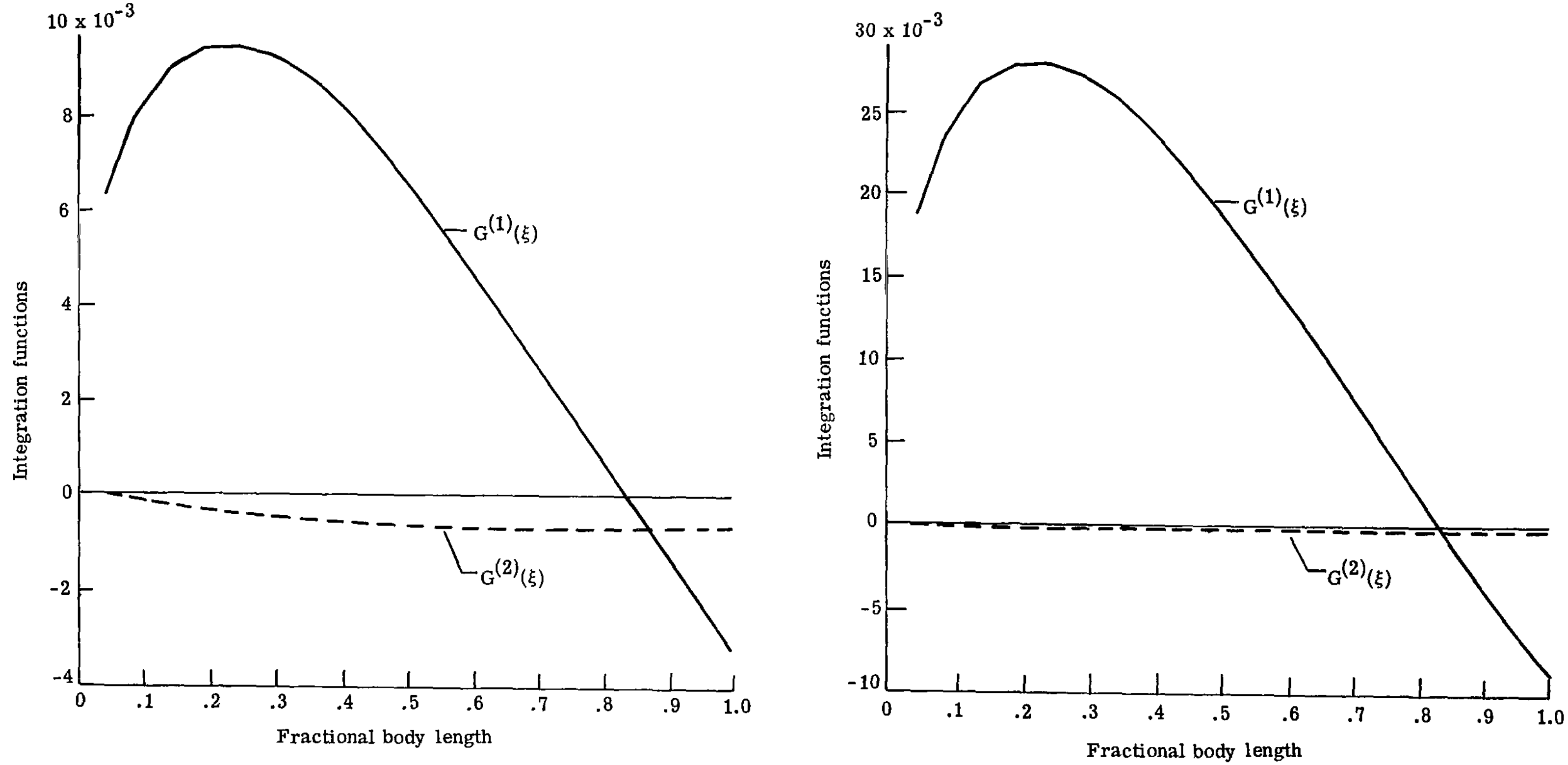
Figure 5.- The shock front and the characteristics.



(a) $A_1 = 0.04\left(x^2 - 2x^3 + x^4\right)\pi$; $M = 1.1$.

(b) $A_1 = 0.04\left(x^2 - 2x^3 + x^4\right)\pi$; $M = 2.5$.

Figure 6.- Comparison of the functions of integration $G^{(1)}(\xi)$ and $G^{(2)}(\xi)$. $R = 20$; $\alpha \cos \psi = 0.1$.



(c) $A_2 = 0.01\left(4x^2 - 4x^3 + x^4\right)\pi$; M = 1.1.

(d) $A_2 = 0.01\left(4x^2 - 4x^3 + x^4\right)\pi$; M = 2.5.

Figure 6.- Concluded.

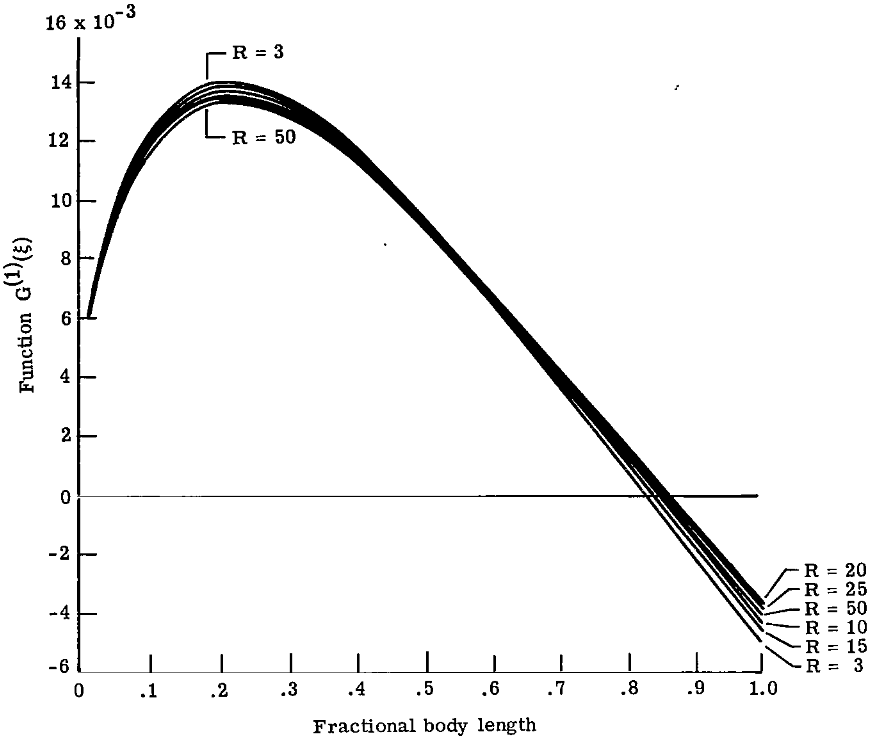


Figure 7.- Influence of the matching parameter R on the function of integration $G^{(1)}(\xi)$. $A_2 = 0.01\left(4x^2 - 4x^3 + x^4\right)\pi$; $M = 1.5$; $\alpha \cos \psi = 0$.

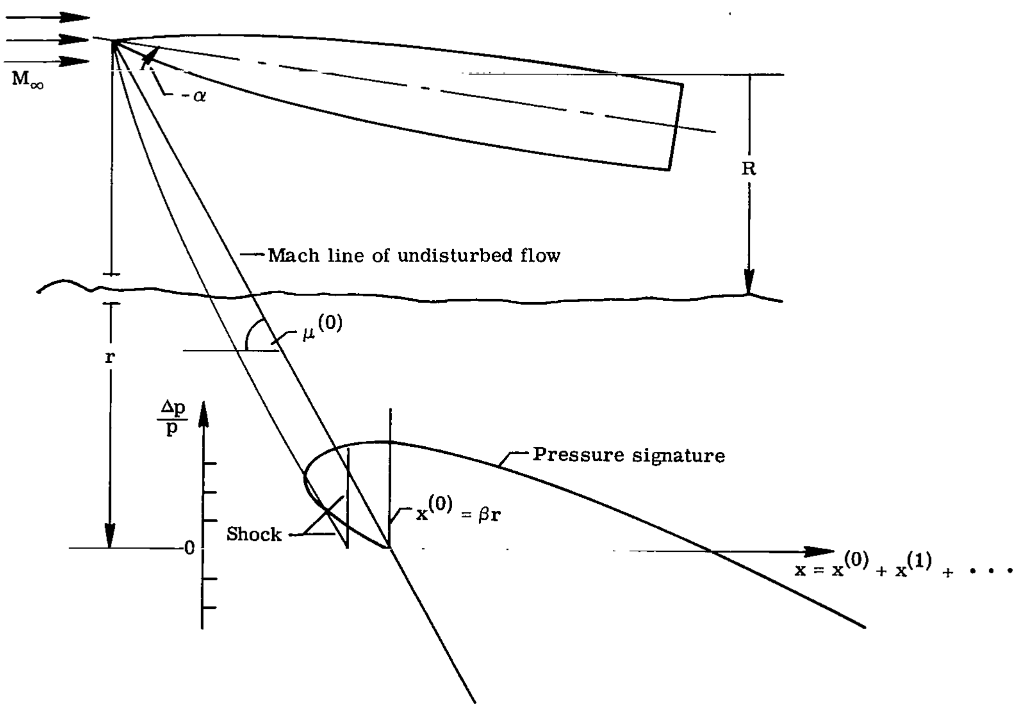
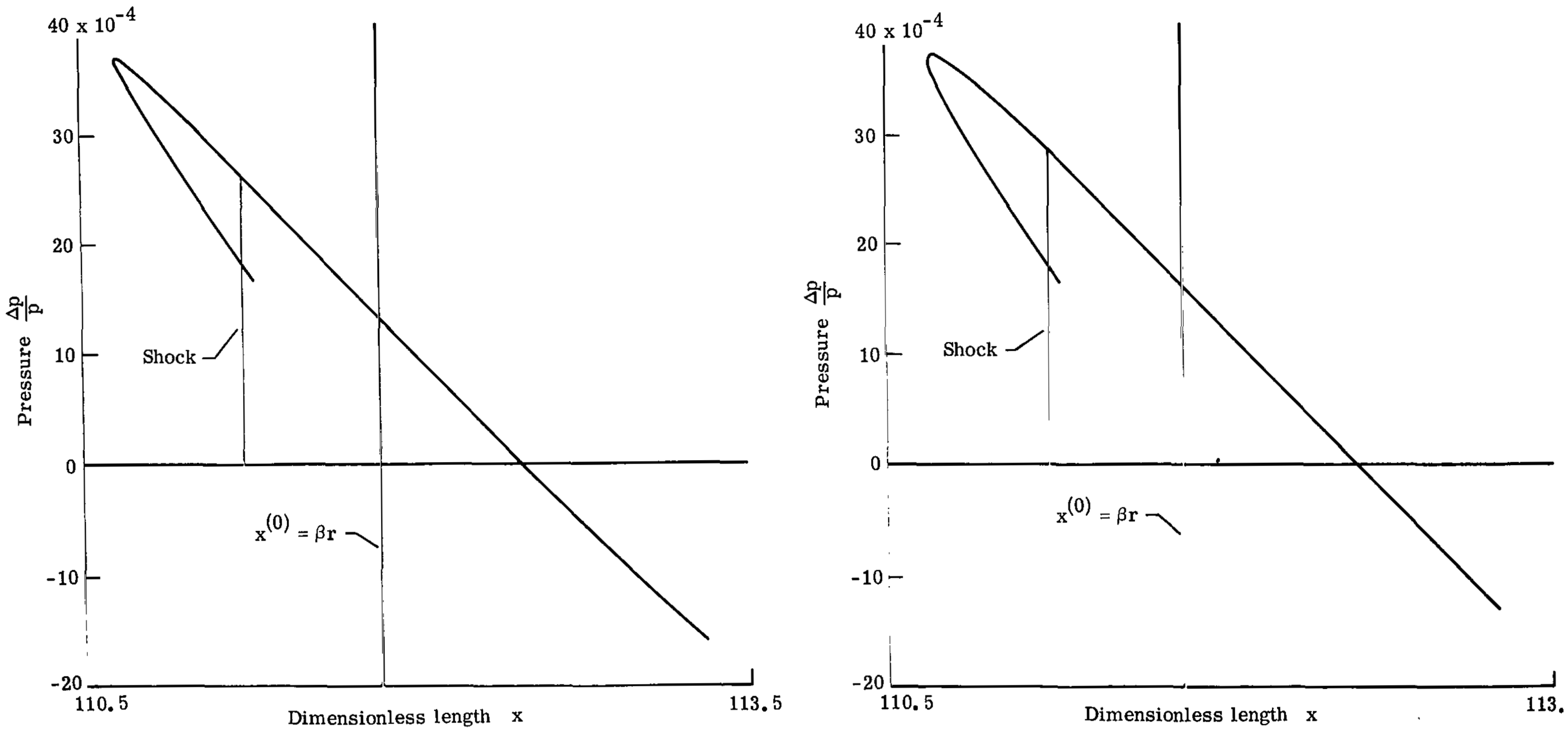


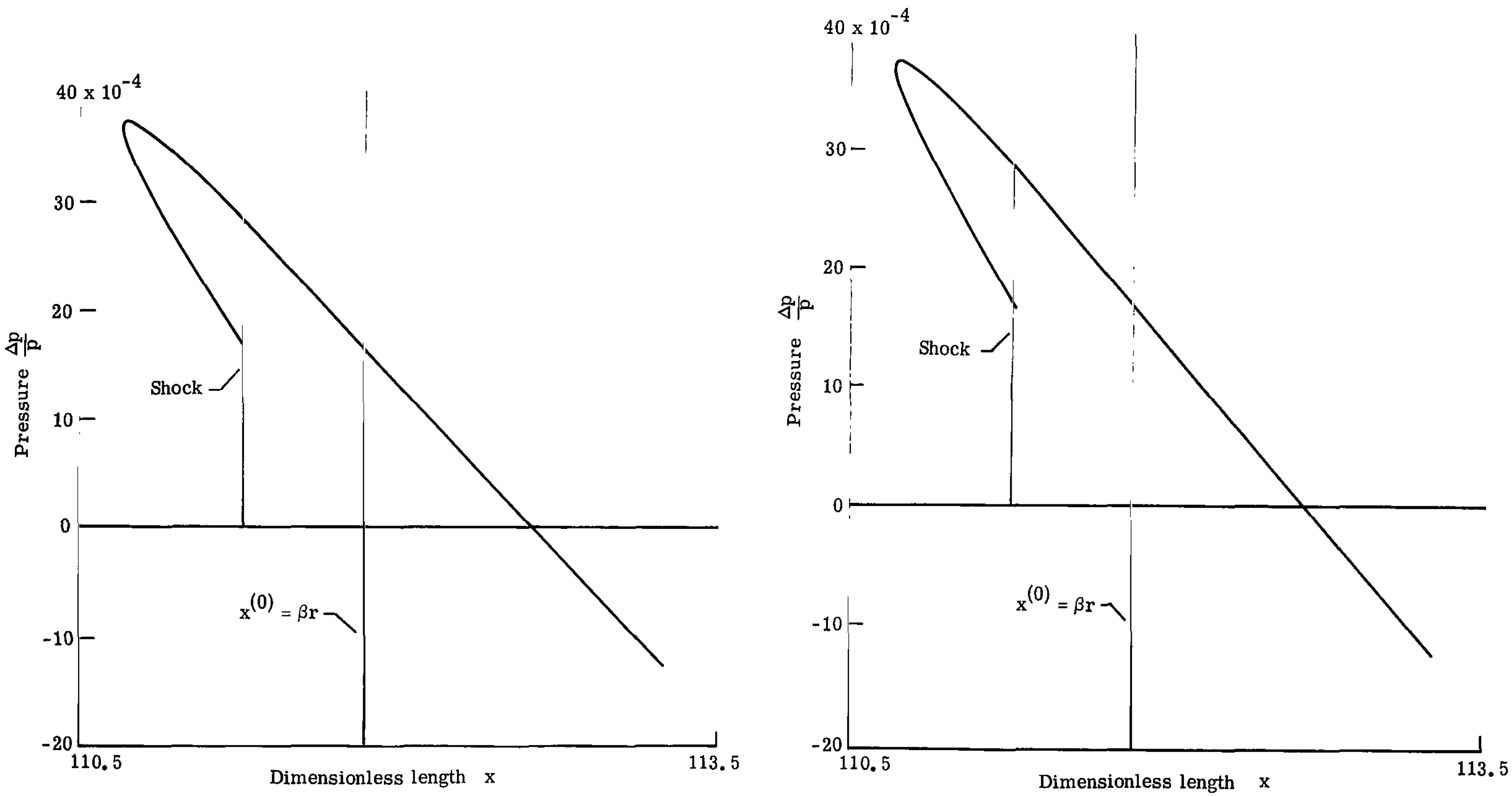
Figure 8.- Description of the construction of the plots of the pressure signature.



(a) R = 3.

(b) R = 10.

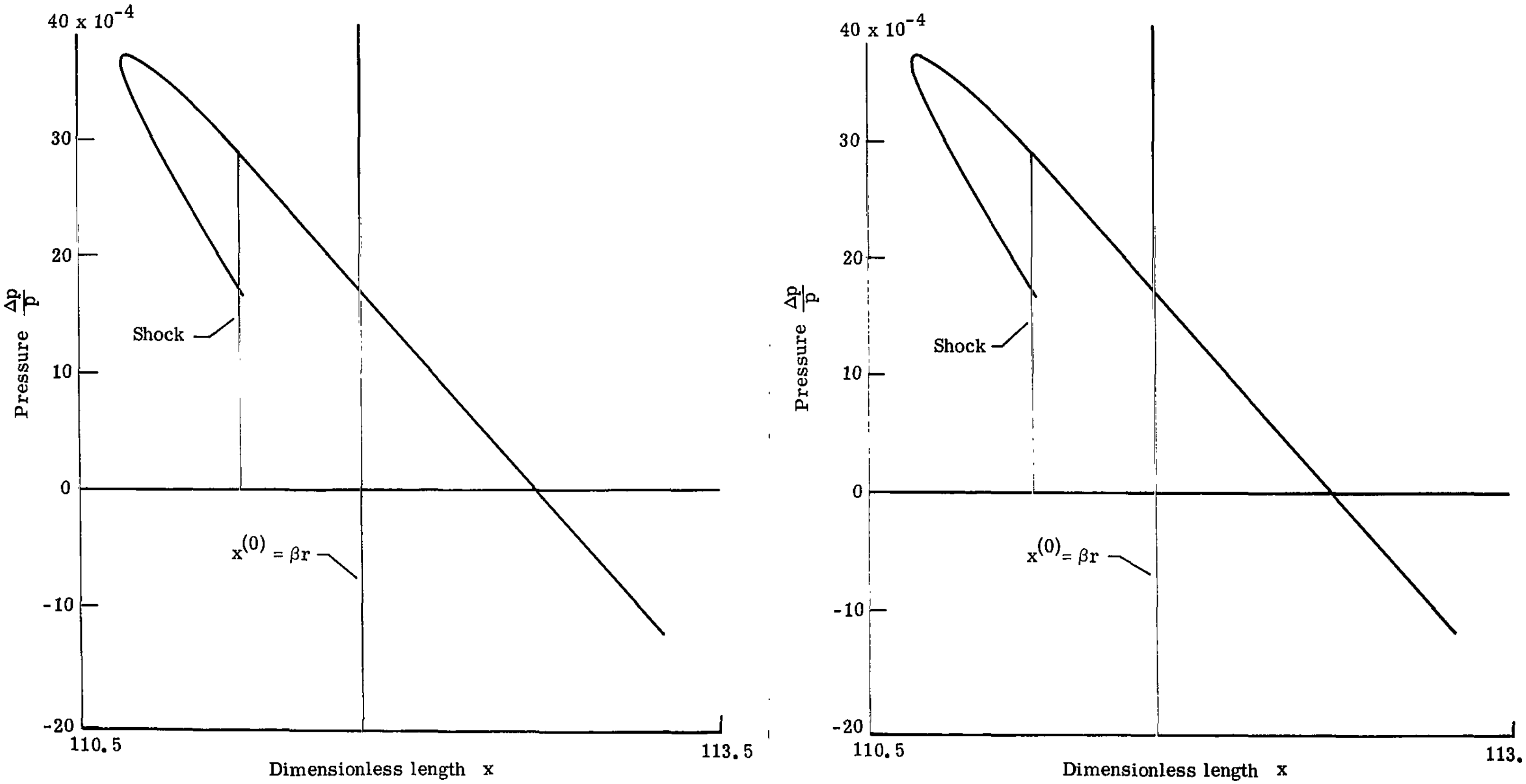
Figure 9.- Influence of the matching parameter R on the pressure signature.
$A_2 = 0.01\left(4x^2 - 4x^3 + x^4\right)\pi$; $M = 1.5$; $r = 100$; $\alpha \cos \psi = 0$.



(c) R = 15.

(d) R = 20.

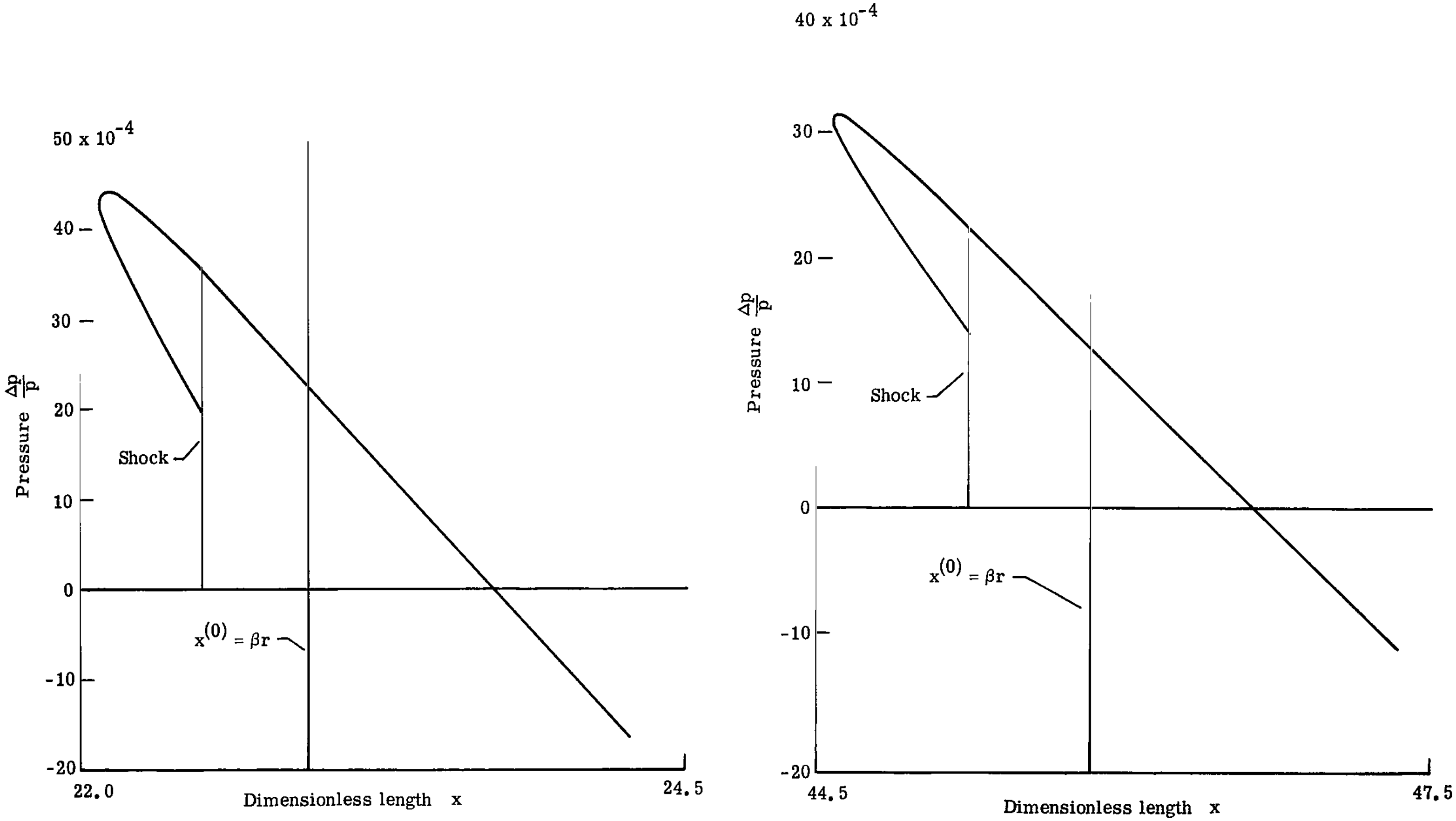
Figure 9.- Continued.



(e) R = 25.

(f) R = 50.

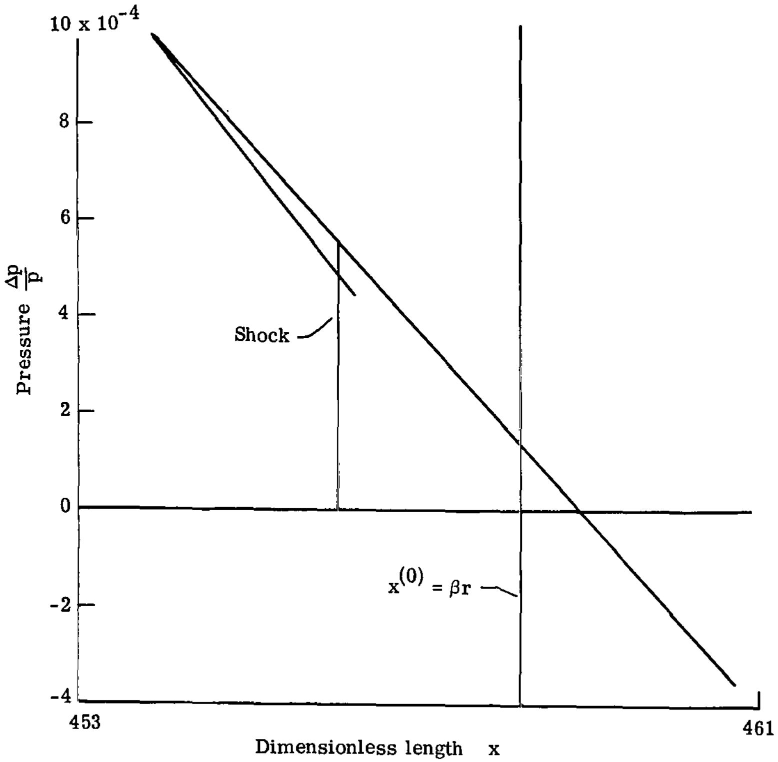
Figure 9.- Concluded.



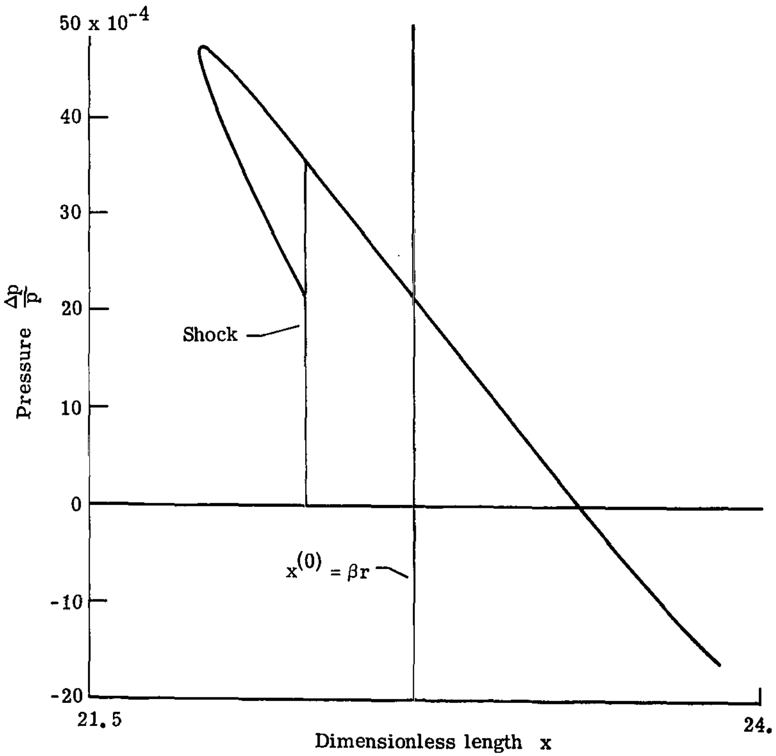
(a) $\alpha \cos \psi = 0$; $r = 50$.

(b) $\alpha \cos \psi = 0$; $r = 100$.

Figure 10.- Influence of the distance r on the pressure signature for $A_2 = 0.01\left(4x^2 - 4x^3 + x^4\right)\pi$. $M = 1.1$; $R = 20$.

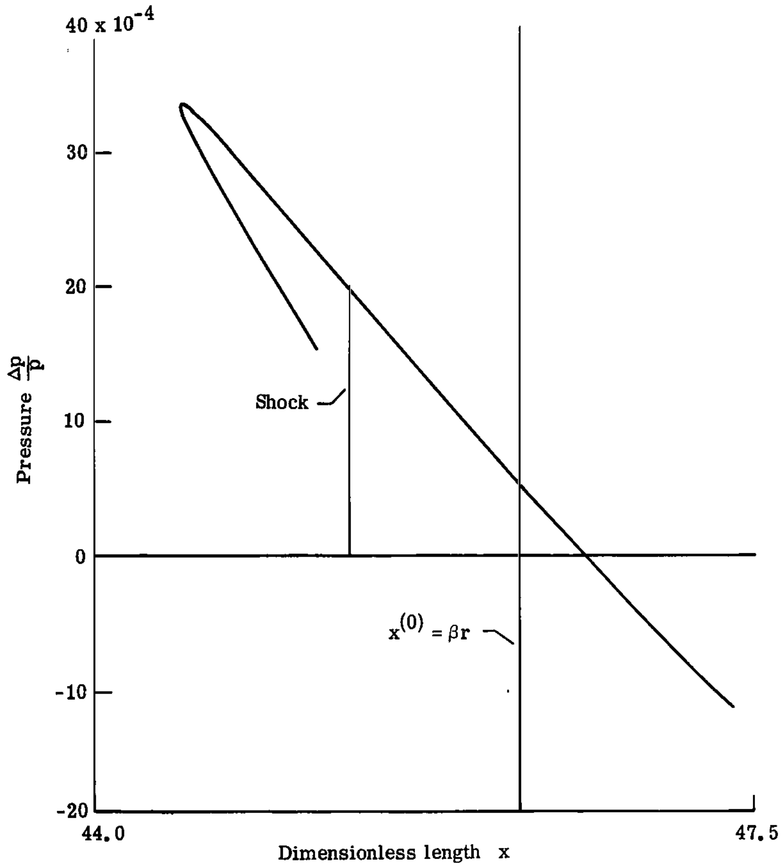


(c) $\alpha \cos \psi = 0$; $r = 1000$.

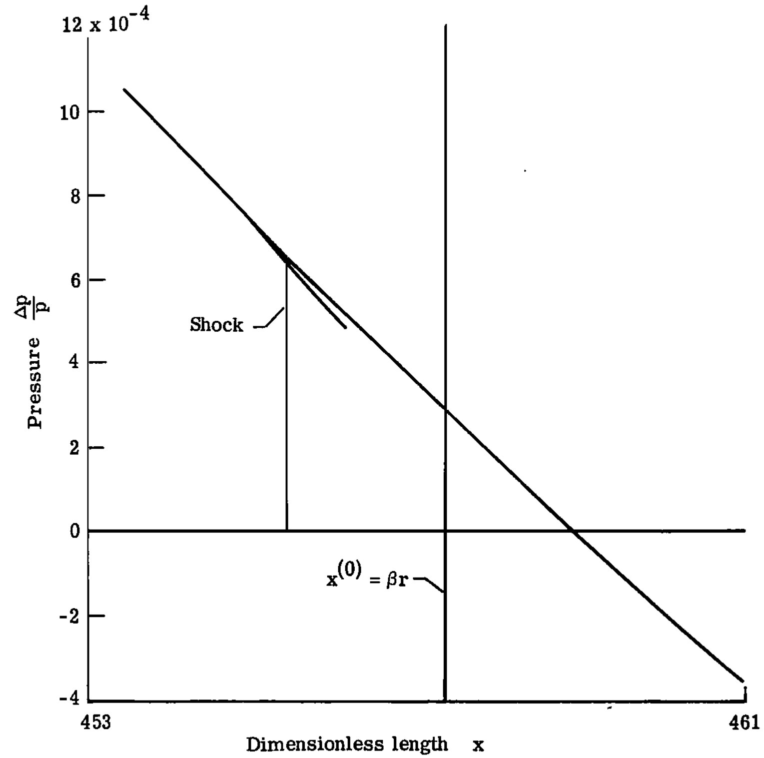


(d) $\alpha \cos \psi = 0.1$; $r = 50$.

Figure 10.- Continued.

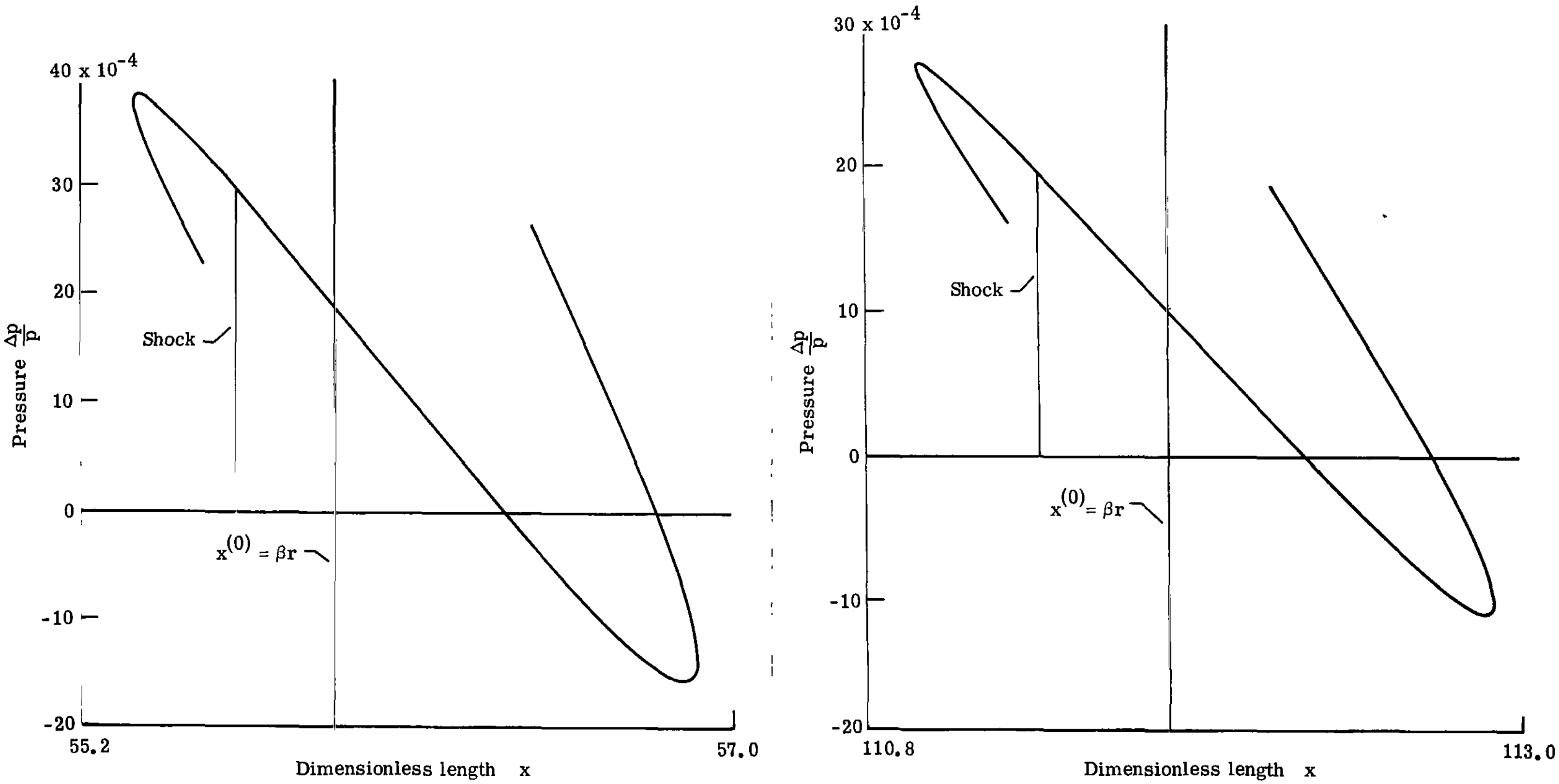


(e) $\alpha \cos \psi = 0.1$; $r = 100.$



(f) $\alpha \cos \psi = 0.1$; $r = 1000.$

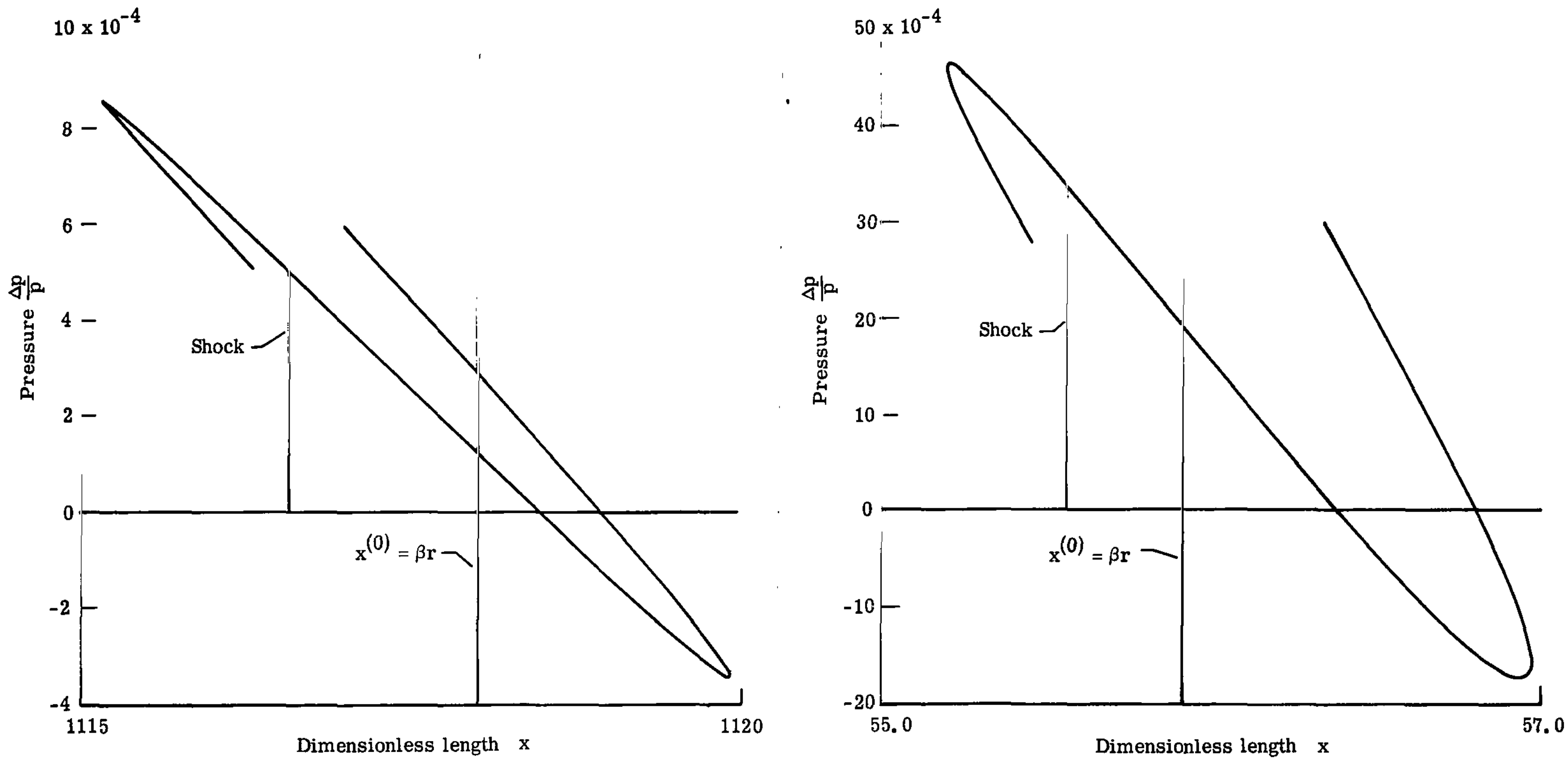
Figure 10.- Concluded.



(a) $\alpha \cos \psi = 0$; $r = 50$.

(b) $\alpha \cos \psi = 0$; $r = 100$.

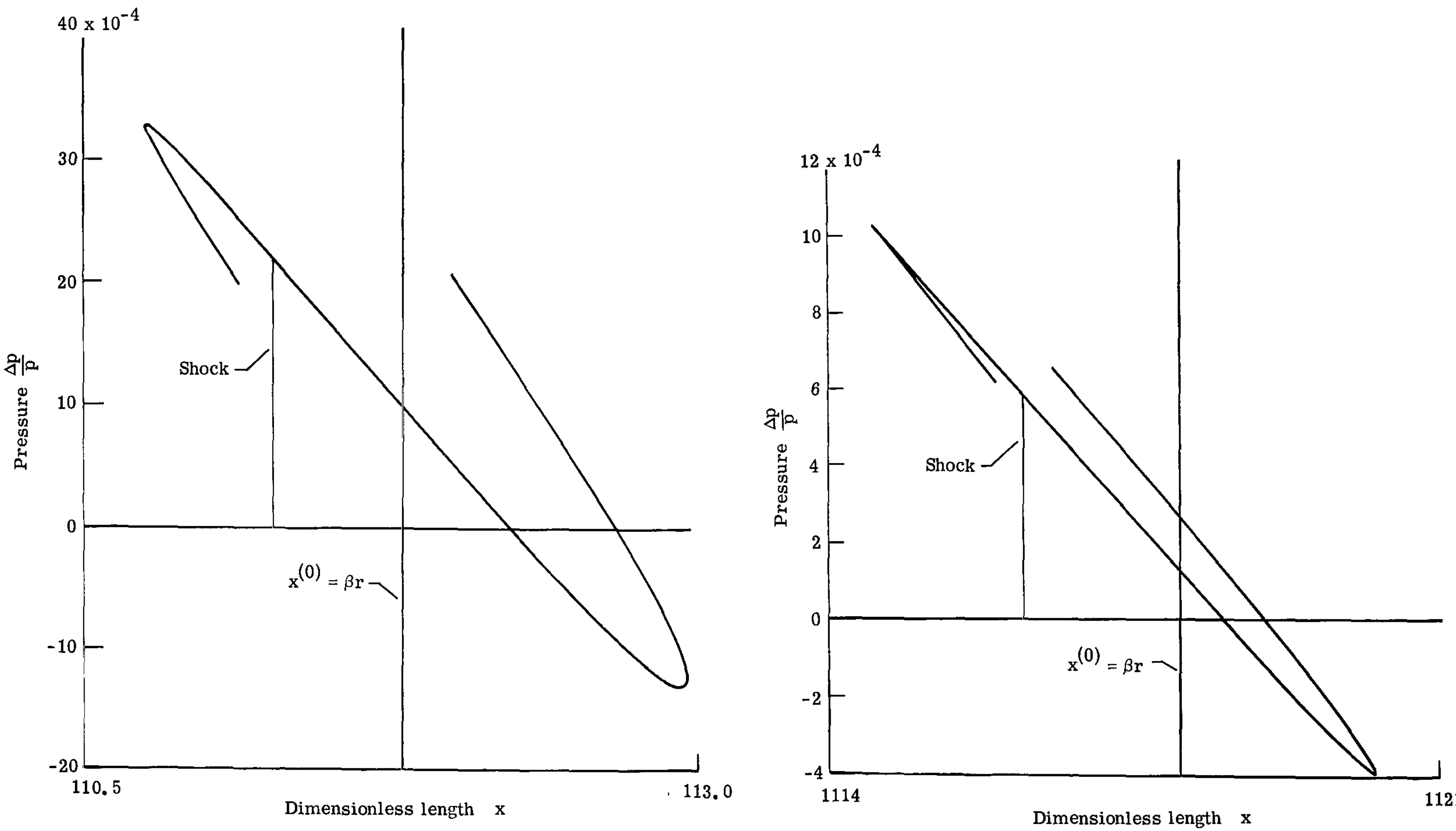
Figure 11.- Influence of the distance r on the pressure signature for $A_1 = 0.04\left(x^2 - 2x^3 + x^4\right)\pi$. $M = 1.5$; $R = 20$.



(c) $\alpha \cos \psi = 0$; $r = 1000$.

(d) $\alpha \cos \psi = 0.1$; $r = 50$.

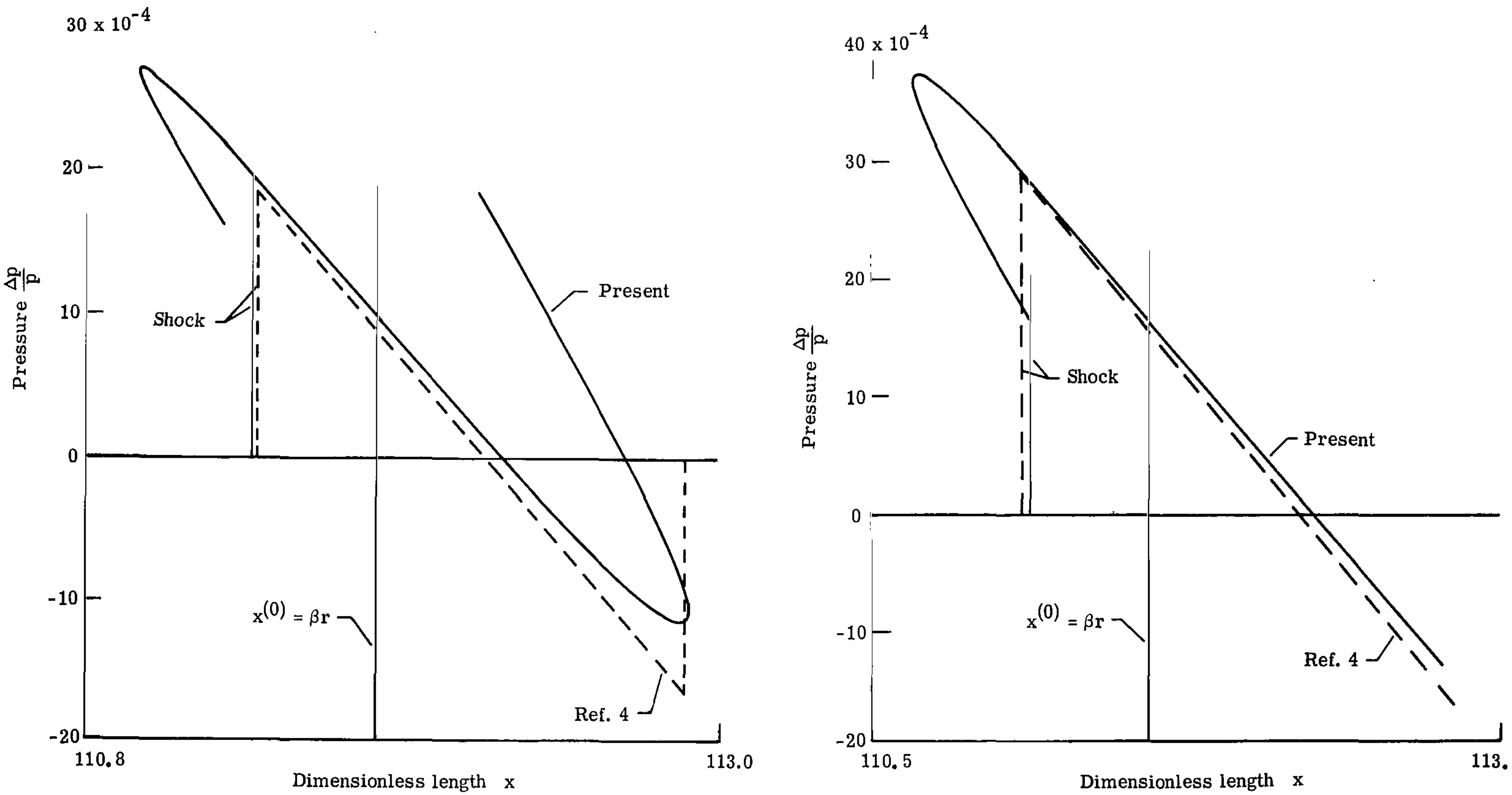
Figure 11.- Continued.



(e) $\alpha \cos \psi = 0.1$; $r = 100$.

(f) $\alpha \cos \psi = 0.1$; $r = 1000$.

Figure 11.- Concluded.



(a) $A_1 = 0.04\left(x^2 - 2x^3 + x^4\right)\pi.$

(b) $A_2 = 0.01\left(4x^2 - 4x^3 + x^4\right)\pi.$

Figure 12.- Comparison of present theory with theory of reference 4. $M = 1.5$; $R = 20$; $r = 100$; $\alpha \cos \psi = 0$.

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